AUSTRALIAN NATIONAL UNIVERSITY
Department of Engineering

ENGN6612/4612 Digital Signal Processing and Control
Problem Set #7 Fast Fourier Transform (FFT)

Q1

A discrete signal $x[n]$ is defined as:

(a)

$$x[n] = \begin{cases} 1 & \text{for } n = 4l+2 \\ 0 & \text{elsewhere} \end{cases}$$

with $l = 0, \pm 1, \pm 2, \cdots$.

(b)

$$x[n] = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1,3 \\ 2 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

For each $x[n]$:

- State whether the signal is periodic, (non-periodic) finite or (non-periodic) finite duration.
- Calculate the 8-point DFT of $x[n]$.
- Assuming $x[n]$ is a finite duration signal (that exists only for $0 \leq n \leq 8$), calculate the DTFT of $x[n]$.
- Show that DFT is sampled version of DTFT (consider both real and imaginary parts).

Q2

Show that the FFT shown schematically in the figure below corresponds to a 4-point DFT. `(challenge problem)`

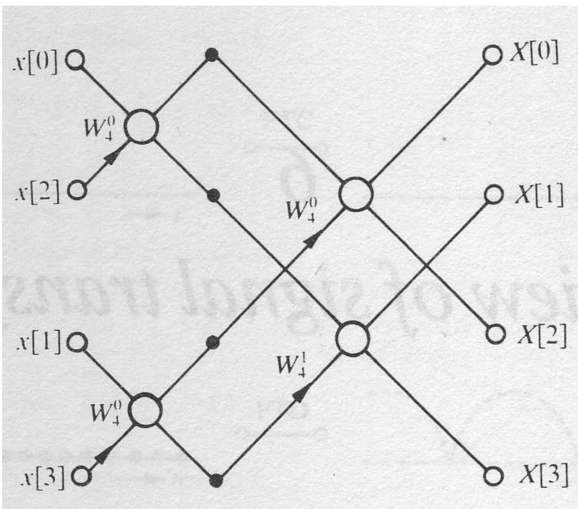

Figure 1: Question 2

Q3

A discrete signal $x[n]$ is defined as:
(a)

$$x[n] = \begin{cases} 1 & \text{for } n = 4l+2 \\ 0 & \text{elsewhere} \end{cases}$$

with $l = 0, \pm 1, \pm 2, \cdots$.

(b)

$$x[n] = \begin{cases} n+1 & \text{for } 0 \le n < 4 \\ 0 & \text{elsewhere} \end{cases}$$

For this signal:

- Calculate $X[k]$ using definition of DFT (take $N = 4$).
- Calculate $X[k]$ by making use of the diagram shown in Question 2.

Q4

Consider the periodic sequences $x_p[n]$ and $h_p[n]$ (with period $N = 4$):

(a)

$$h_p[n] = \begin{cases} n+1 & \text{for } 0 \le n \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$x_p[n] = \begin{cases} 1 & \text{for } n = 1, 2 \\ 0 & \text{elsewhere} \end{cases}$$

(b)

$$h_p[n] = \begin{cases} n & \text{for } 0 \le n \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$x_p[n] = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

Determine the output $z_p[n] = x_p[n] \circledast h_p[n]$ using both (i) graphical discrete-time circular convolution and (ii) DFT method.

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Problem Set #7 Solution

Q1

(a) Complete Solution

The given signal is periodic with period $N = 4$.
The plot of first 8 samples of the signal is shown below:-

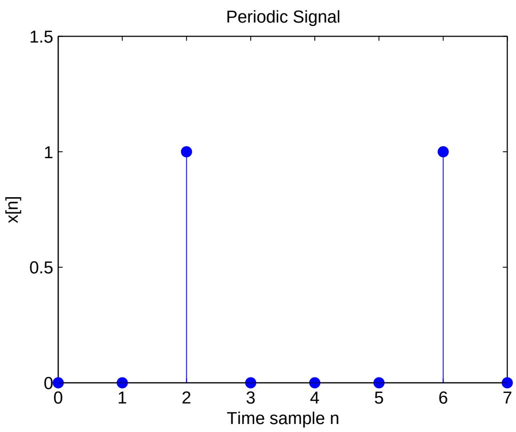


Figure 2: Question 1(a)

DFT

The 8-point DFT of $x[n]$ is

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{7} x[n]e^{-j\frac{\pi}{4}kn} \\ &= x[2]e^{-j\frac{\pi}{2}k} + x[6]e^{-j\frac{3\pi}{2}k} \\ &= e^{-j\frac{\pi}{2}k} + e^{-j\frac{3\pi}{2}k} \end{aligned}$$

Hence $X[0] = 2$, $X[1] = 0$, $X[2] = -2$, $X[3] = 0$, $X[4] = 2$, $X[5] = 0$, $X[6] = -2$, $X[7] = 0$.

The result is summarised in the table below:-

Frequency Sample k	Discrete Frequency $\omega_k = \frac{2\pi k}{N}$ (rad/s)	$\Re\{X[k]\}$	$\Im\{X[k]\}$
0	0	2	0
1	$\frac{\pi}{4}$	0	0
2	$\frac{\pi}{2}$	-2	0
3	$\frac{3\pi}{4}$	0	0
4	π	2	0
5	$\frac{5\pi}{4}$	0	0
6	$\frac{3\pi}{2}$	-2	0
7	2π	0	0

DTFT

Assuming $x[n]$ is a finite duration signal (that exists only for $0 \le n \le 8$), we have

$$\begin{aligned} x[n] &= \delta[n-2]+\delta[n-6] \\ X(z) &= \frac{1}{z^2}+\frac{1}{z^6} \\ X(e^{j\omega}) &= e^{-j2\omega}+e^{-j6\omega} = \{\cos(2\omega)+\cos(6\omega)\}+j\{-\sin(2\omega)-\sin(6\omega)\} \end{aligned}$$

Evaluating $\Re\{X(e^{j\omega})\}$ and $\Im\{X(e^{j\omega})\}$ for the fundamental interval $0 \le \omega \le 2\pi$, we have

Frequency ω (rad/s)	$\Re\{X(e^{j\omega})\} = \cos(2\omega)+\cos(6\omega)$	$\Im\{X(e^{j\omega})\} = -\sin(2\omega)-\sin(6\omega)$
0	2	0
$\frac{\pi}{4}$	0	0
$\frac{\pi}{2}$	-2	0
$\frac{3\pi}{4}$	0	0
π	2	0
$\frac{5\pi}{4}$	0	0
$\frac{3\pi}{2}$	-2	0
2π	0	0

Comparing the results in the two tables, we see that DFT is sampled version of DTFT.

The plots are shown in the figures below:

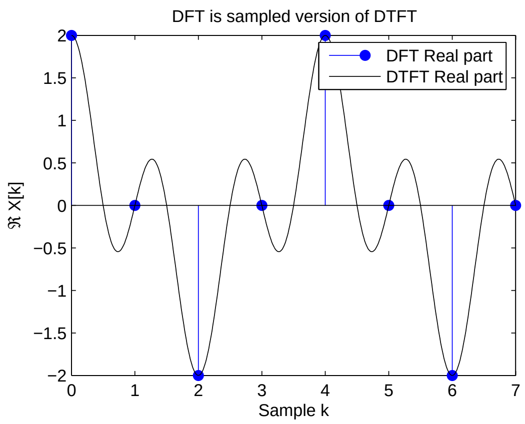


Figure 3: Question 1(a): DFT and DTFT Real part

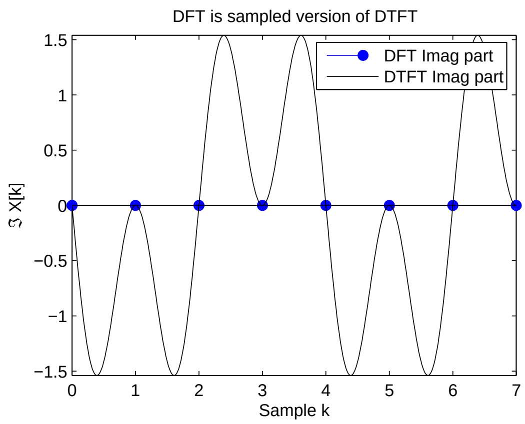


Figure 4: Question 1(a): DFT and DTFT Imaginary part

Compare with 4-point DFT evaluated in `Problem Set 6: Q1a.`

(b) Partial Solution

The given signal is (non-periodic) finite duration.
The plot of first 8 samples of the signal is shown below:-

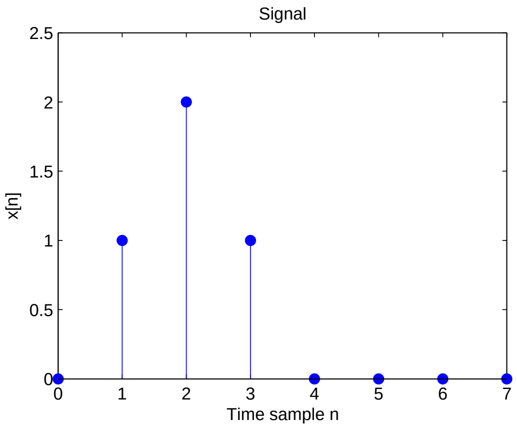


Figure 5: Question 1(b)

DFT

The 8-point DFT of $x[n]$ is

$$X[k] = e^{-j\frac{\pi}{4}k} + 2e^{-j\frac{\pi}{2}k} + e^{-j\frac{3\pi}{4}k}$$

DTFT

$$X(e^{j\omega}) = e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$$

The results are summarised in the tables below:-

Frequency ω (rad/s)	$\Re\{X(e^{j\omega})\}$	$\Im\{X(e^{j\omega})\}$
0	4	0
$\frac{\pi}{4}$	0	-3.4142
$\frac{\pi}{2}$	-2	0
$\frac{3\pi}{4}$	0	0.5858
π	0	0
$\frac{5\pi}{4}$	0	-0.5858
$\frac{3\pi}{2}$	-2	0
2π	0	3.4142

Frequency Sample k	Discrete Frequency $\omega_k = \frac{2\pi k}{N}$ (rad/s)	$\Re\{X[k]\}$	$\Im\{X[k]\}$
0	0	4	0
1	$\frac{\pi}{4}$	0	-3.4142
2	$\frac{\pi}{2}$	-2	0
3	$\frac{3\pi}{4}$	0	0.5858
4	π	0	0
5	$\frac{5\pi}{4}$	0	-0.5858
6	$\frac{3\pi}{2}$	-2	0
7	2π	0	3.4142

The plots are shown in the figures below:

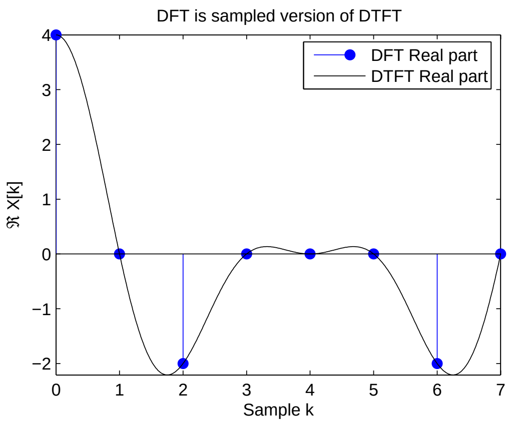


Figure 6: Question 1(b): DFT and DTFT Real part

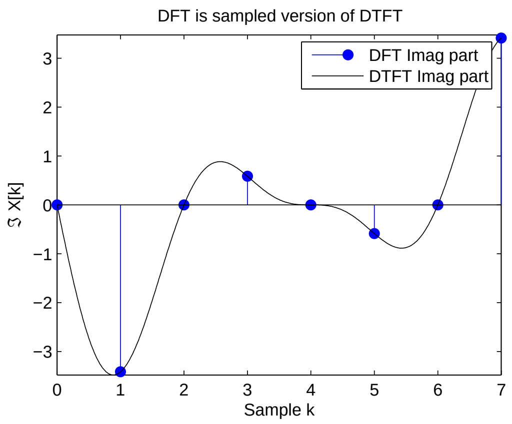


Figure 7: Question 1(b): DFT and DTFT Imaginary part

Q2

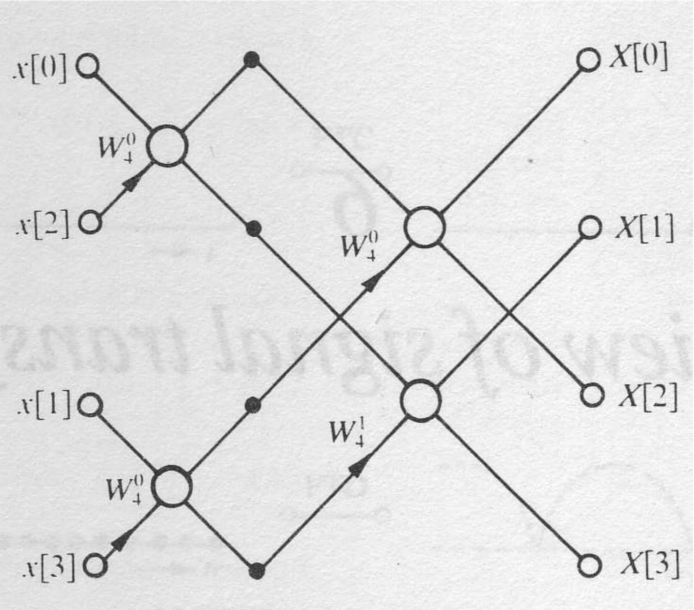


Figure 8: Question 2

The proof is left as an exercise for the students.

Hint

By tracing the paths in the flow graph of Fig. 8, show that each input sample contributes the proper amount to the output of the DFT sample, i.e. verify that

$$\begin{aligned}
X[0] &= \sum_{n=0}^{N-1} x[n] \\
X[1] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}n} \\
X[2] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}2n} \\
X[3] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}3n}
\end{aligned}$$

Reference

Please see Chapter 9 in “Discrete-Time Signal Processing” by Oppenheim and Schafer for comprehensive discussion of FFT.

Q3

(a) Partial Solution

Using DFT Definition

$$\begin{aligned} X[k] &= e^{-j\pi k} \\ X[0] &= e^{-j0} = 1 = 1 + j0 \\ X[1] &= e^{-j\pi} = -1 = -1 + j0 \\ X[2] &= e^{-j2\pi} = 1 = 1 + j0 \\ X[3] &= e^{-j3\pi} = -1 = -1 + j0 \end{aligned}$$

For details, see `Problem Set 06: Q1 (a)`.

Using FFT butterfly

We have the twiddle factor

$$\begin{aligned} W_N^p &= e^{-j\frac{2\pi}{N}p} \\ &= e^{-j\frac{\pi}{2}p} \end{aligned}$$

Hence,

$$\begin{aligned} W_4^0 &= e^{-j0} = 1 \\ W_4^1 &= e^{-j\frac{\pi}{2}} = -j \end{aligned}$$

Writing the equations for the intermediate terms in the diagram, we have

$$\begin{aligned} a &= x[0] + W_4^0 x[2] = x[0] + x[2] \\ b &= x[0] - W_4^0 x[2] = x[0] - x[2] \\ c &= x[1] + W_4^0 x[3] = x[1] + x[3] \\ d &= x[1] - W_4^0 x[3] = x[1] - x[3] \end{aligned}$$

Writing the equations for the output terms in the diagram, we have

$$\begin{aligned} X[0] &= a + W_4^0 c = a + c \\ X[2] &= a - W_4^0 c = a - c \\ X[1] &= b + W_4^1 d = b - jd \\ X[3] &= b - W_4^1 d = b + jd \end{aligned}$$

The output samples $X[k]$ can be expressed in terms of input samples $x[n]$ as

$$\begin{aligned} X[0] &= \{x[0] + x[2]\} + \{x[1] + x[3]\} \\ X[2] &= \{x[0] + x[2]\} - \{x[1] + x[3]\} \\ X[1] &= \{x[0] - x[2]\} - j\{x[1] - x[3]\} \\ X[3] &= \{x[0] - x[2]\} + j\{x[1] - x[3]\} \end{aligned}$$

Substituting the values,
$X[0] = 1$, $X[1] = -1$, $X[2] = 1$, $X[3] = -1$.

(b) Solution

$X[0] = 10$, $X[1] = -2 + j2$, $X[2] = -2$, $X[3] = -2 - j2$.

Check answer in Matlab using the following commands

```
>> n=[0 1 2 3];
>> x=[1 2 3 4];
>> X=fft(x);
```

Q4

(a) Complete Solution

Please see attached pages 10 – 12.

(b) Solution

The input sequences are shown in the figure below:

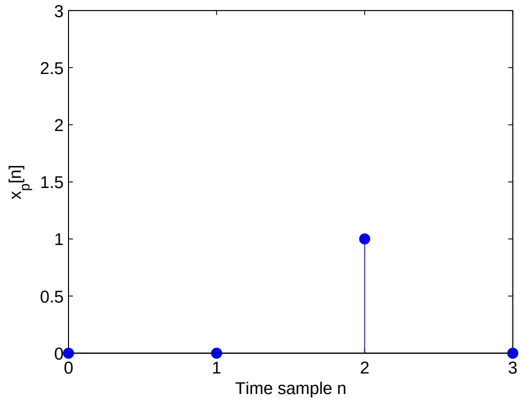


Figure 9: Question 4(b): Periodic sequence $x_p[n]$.

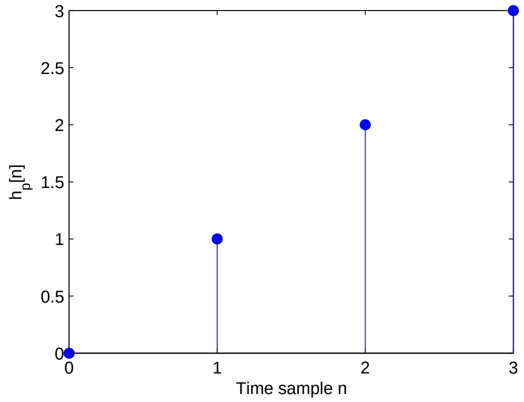


Figure 10: Question 4(b): Periodic sequence $h_p[n]$.

The output $z_p[n] = x_p[n] \circledast h_p[n]$ is shown in the figure below:

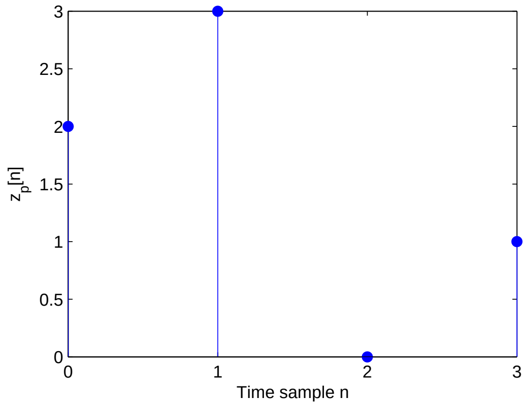


Figure 11: Question 4(b): Output periodic sequence $z_p[n]$.

Check answer in Matlab using the following commands

```
>> xp=[0 0 1 0];
>> hp=[0 1 2 3];
>> zp=ifft(fft(xp).*fft(hp))
```

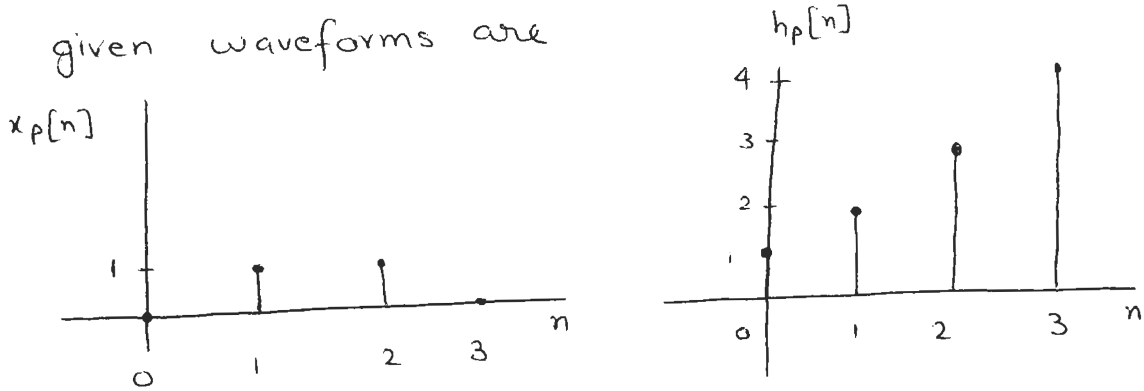
PROBLEM SET 07 : Q4 (a)

GRAPHICAL DISCRETE-TIME CIRCULAR CONVOLUTION

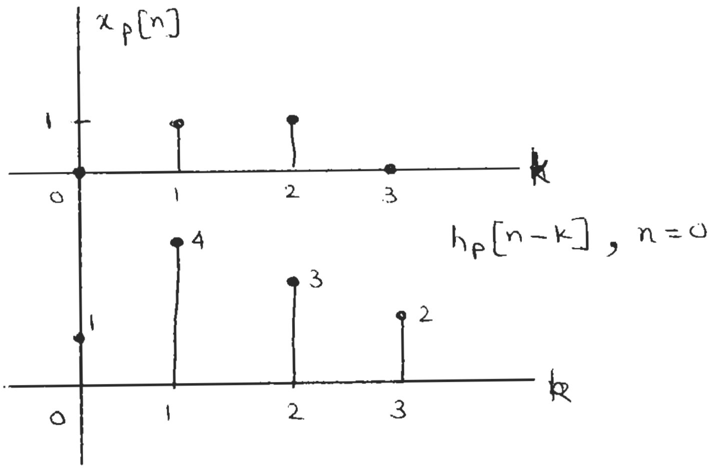
We know the circular convolution of two periodic signals with Period N is defined as

$$z_p[n] = \sum_{k=0}^{N-1} x_p[k]\, h_p[n-k]$$

The given waveforms are



n = 0



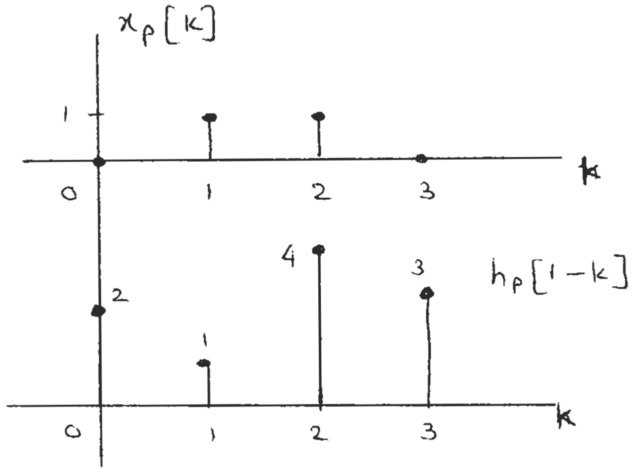
$$z_p[0] = \sum_{k=0}^{3} x_p[k]\, h_p[0-k]$$

$$= (4)(1) + (3)(1)$$

$$= 7.$$

(Please note technique for circular reversal shown in the figure).

$n = 1$



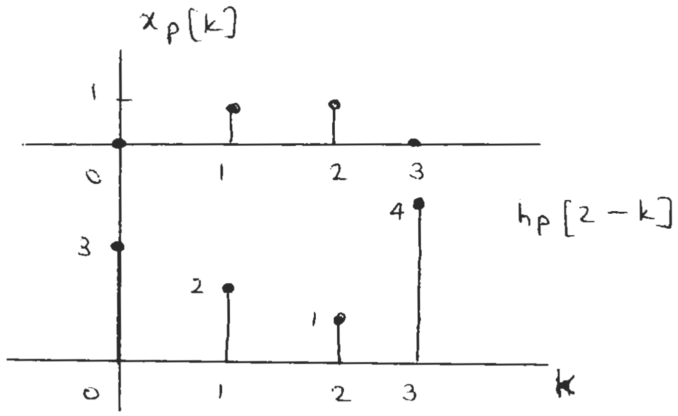
$$z_p[1] = \sum_{k=0}^{3} x_p[k]\, h_p[1-k]$$

(Please note technique of circular time shift shown in figures)

$$= (1)(1) + (4)(1)$$

$$= 5$$

$n = 2$

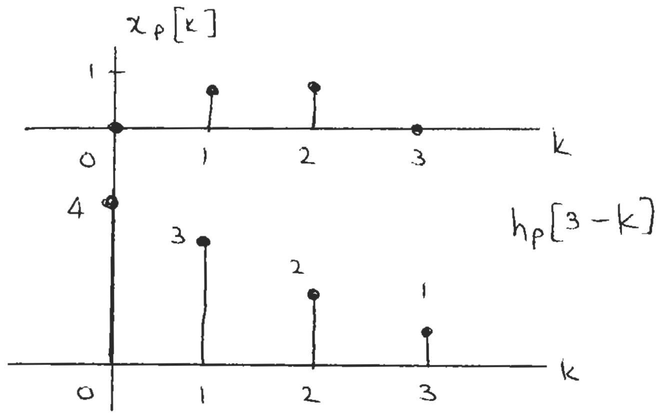


$$z_p[2] = \sum_{k=0}^{3} x_p[k]\, h_p[2-k]$$

$$= (2)(1) + (1)(1)$$

$$= 3$$

$n = 3$

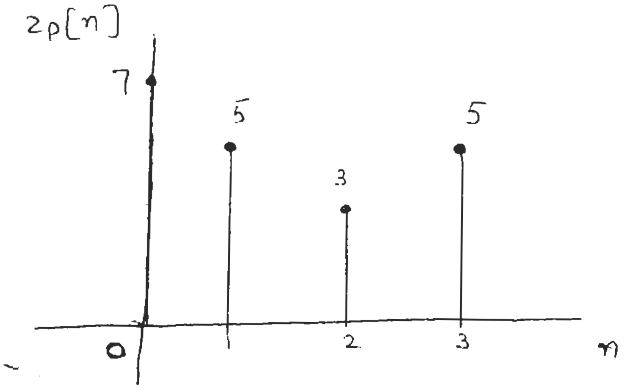


$$z_p[3] = \sum_{k=0}^{4} x_p[k]\, h_p[3-k]$$

$$= (3)(1) + (2)(1)$$

$$= 5$$

The output waveform is shown below



FREQUENCY DOMAIN

Show that 4-point DFT of $x_p[n]$ is

$$X_p[k] = e^{-j\pi k}$$

Hence

$X_p[0] = 1$

$X_p[1] = -1$

$X_p[2] = 1$

$X_p[3] = -1$

Show that 4-point DFT of $h_p[n]$ is

$H_p[0] = 6$

$H_p[1] = -2+j2$

$H_p[2] = -2$

$H_p[3] = -2-j2$

(Hint :- 4-point FFT with FFT butterfly equations can also be used to calculate this result)

For output

$$Z_p[k] = X_p[k] \, H_p[k]$$

Hence $Z_p[0] = 6$

$Z_p[1] = 2-j2$

$Z_p[2] = -2$

$Z_p[3] = 2+j2.$

Take 4-point IDFT of $Z_p[k]$ and show that $Z_p[n]$ is

$Z_p[0] = 7$, $z_p[1] = 5$, $Z_p[2] = 3$, $z_p[3] = 5$