**AUSTRALIAN NATIONAL UNIVERSITY**
**Department of Engineering**

# ENGN6612/4612 Digital Signal Processing and Control
# Problem Set #5 Discrete Time Fourier Transform (DTFT)

## Q1

Find $X(e^{j\omega})$ and sketch $|X(e^{j\omega})|$ and $\angle X(e^{j\omega})$ when $x[n]$ is given by the following:
(a) $a^n u[n], \qquad a = -0.6$
(b) $\delta[n-3]$ `(challenge problem)`

## Q2

Consider a discrete time filter described by the following difference equations:
(a) $y[n] = \frac{1}{2}(x[n] + x[n-1])$ (This is called a two-point moving-average filter)
(b) $y[n] = x[n] + 0.6y[n-1]$
(c) $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$ `(challenge problem)`.

For each filter:-

- Find the transfer function $H(z)$.
- Find whether the filter is FIR or IIR.
- Determine the frequency response $H(e^{j\omega})$ .
- Determine and roughly sketch magnitude of the frequency response of the filter for $-\pi \leq \omega \leq \pi$.

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# Problem Set #5 Solution

# Q1

## (a) Partial Solution

Given that

$$x[n] = a^n u[n], \qquad a = -0.6$$

Taking the $z$-transform , we have

$$X(z) = \frac{z}{z-a}$$

Let

$$z = e^{j\omega}$$

Hence the frequency response is given by

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}$$

**Magnitude Response**

We have

$$X(e^{j\omega}) = \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega}$$

The magnitude response is given by

$$\begin{aligned}|X(e^{j\omega})| &= \frac{|\cos\omega + j\sin\omega|}{|(\cos\omega - a) + j\sin\omega|} \\ &= \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}\end{aligned}$$

Evaluating $|X(e^{j\omega})|$ for $-\pi \leq \omega \leq \pi$, we have

| $\omega$ (rad/s) | $\lvert X(e^{j\omega})\rvert$ |
|---|---|
| $-\pi$ | 2.5000 |
| -3 | 2.4112 |
| -2 | 1.0779 |
| $-\pi/2$ | 0.8575 |
| -1 | 0.7056 |
| 0 | 0.6250 |
| 1 | 0.7056 |
| $\pi/2$ | 0.8575 |
| 2 | 1.0779 |
| 3 | 2.4112 |
| $\pi$ | 2.5000 |

The plot of $|X(e^{j\omega})|$ is shown in the figure below:

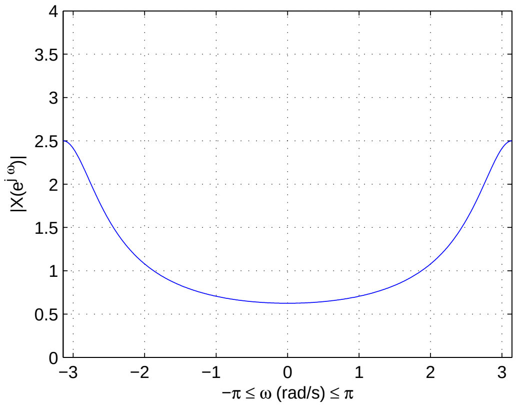



Figure 1: Question 1(a)

**Phase Response**

We have

$$X(e^{j\omega}) = \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega}$$
$$= \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega} \times \frac{(\cos\omega - a) - j\sin\omega}{(\cos\omega - a) - j\sin\omega}$$

Show that the above expression simplifies to

$$X(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} + j\frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}$$

The phase response is given by

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{\Im\{H(e^{j\omega})\}}{\Re\{H(e^{j\omega})\}}\right)$$
$$= \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$

Evaluating $\angle X(e^{j\omega})$ for $-\pi \leq \omega \leq \pi$, we have

| $\omega$ (rad/s) | $\angle X(e^{j\omega})$ (degs) |
|---|---|
| $-\pi$ | 0 |
| -3 | -11.7802 |
| -2 | -36.0222 |
| $-\pi/2$ | -30.9638 |
| -1 | -20.8708 |
| 0 | 0 |
| 1 | 20.8708 |
| $\pi/2$ | 30.9638 |
| 2 | 36.0222 |
| 3 | 11.7802 |
| $\pi$ | 0 |

The plot of $\angle X(e^{j\omega})$ is shown in the figure below:

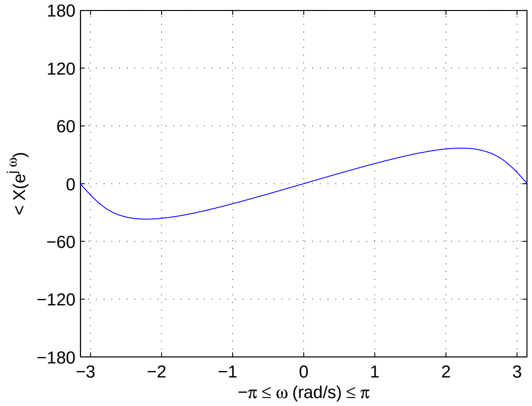



Figure 2: Question 1(a)

## (b) Solution

The frequency response is given by

$$X(e^{j\omega}) \quad = \quad e^{-j3\omega}$$

The plot of $|X(e^{j\omega})|$ is shown in the figure below:

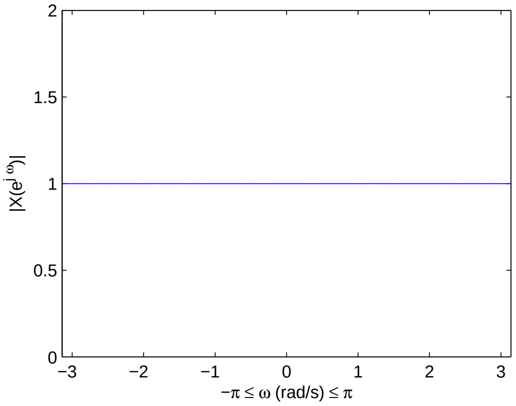



Figure 3: Question 1(b)

The plot of $\angle X(e^{j\omega})$ is shown in the figure below:

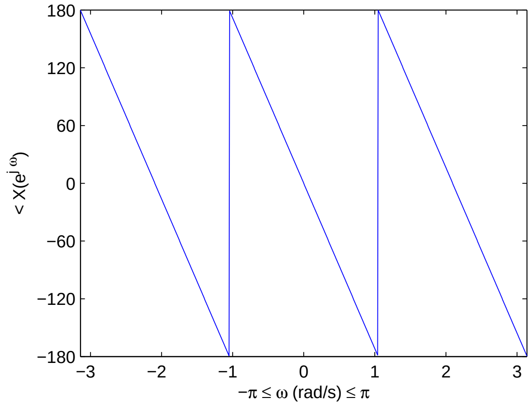



Figure 4: Question 1(b)

# Q2

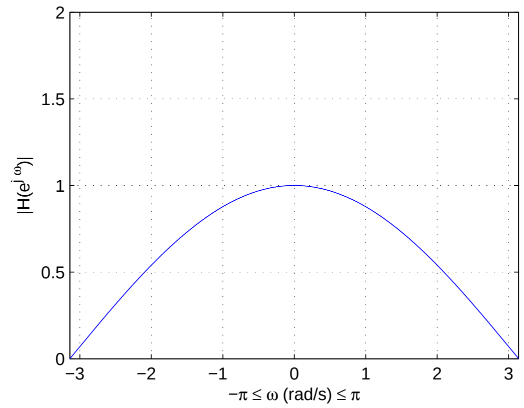
## (a) Partial Solution

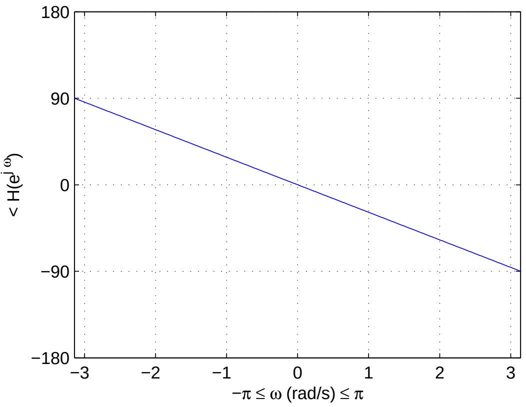
$$\begin{aligned}
H(z) &= \frac{1}{2}+\frac{1}{2}z^{-1} \\
h[n] &= \frac{1}{2}\delta[n]+\frac{1}{2}\delta[n-1] \qquad \text{(FIR filter)} \\
H(e^{j\omega}) &= \frac{1}{2}(1+e^{-j\omega}) \\
&= \frac{1}{2}e^{-j\omega/2}(e^{j\omega/2}+e^{-j\omega/2}) \\
&= e^{-j\omega/2}\cos(\omega/2)
\end{aligned}$$

Evaluating $|H(e^{j\omega})|$ for $-\pi \le \omega \le \pi$, we have

| ω (rad/s) | $\|H(e^{j\omega})\|$ |
|---|---|
| $-\pi$ | 0.0000 |
| -3 | 0.0707 |
| -2 | 0.5403 |
| -1 | 0.8776 |
| 0 | 1.0000 |
| 1 | 0.8776 |
| 2 | 0.5403 |
| 3 | 0.0707 |
| $\pi$ | 0.0000 |

The plots are shown below:-



## (b) Solution

$$
\begin{aligned}
H(z) &= \frac{1}{1-0.6z^{-1}} \\
h[n] &= (0.6)^n u[n] \qquad \text{(IIR filter)} \\
H(e^{j\omega}) &= \frac{1}{1-0.6e^{-j\omega}}
\end{aligned}
$$

Evaluating $|H(e^{j\omega})|$ for $-\pi \leq \omega \leq \pi$, we have

| $\omega$ (rad/s) | $\lvert H(e^{j\omega})\rvert$ |
|---|---|
| $-\pi$ | 0.6250 |
| -3 | 0.6265 |
| -2 | 0.7334 |
| -1 | 1.1854 |
| 0 | 2.5000 |
| 1 | 1.1854 |
| 2 | 0.7334 |
| 3 | 0.6265 |
| $\pi$ | 0.6250 |

The plot is shown below:-

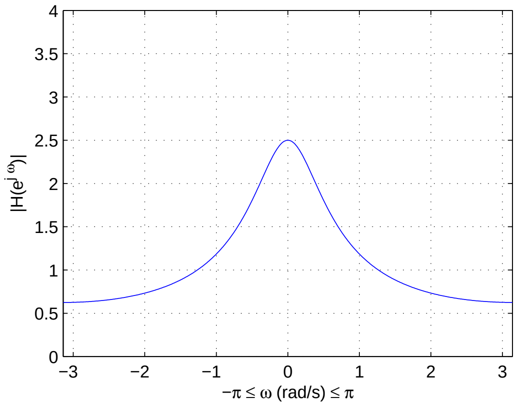



Figure 5: Question 2(b)

## (c) Solution

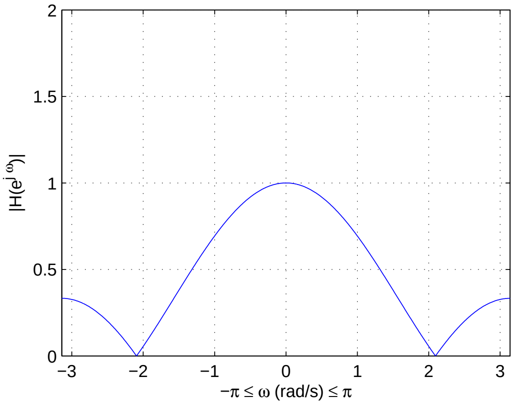
This is a three-point moving-average filter.

$$
\begin{aligned}
H(z) &= \frac{1}{3}(z+1+z^{-1}) = \frac{1}{3}\frac{1+z^{-1}+z^{-2}}{z^{-1}} \\
h[n] &= \frac{1}{3}(\delta[n+1]+\delta[n]+\delta[n-1]) \qquad \text{(FIR filter)} \\
H(e^{j\omega}) &= \frac{1}{3}(e^{j\omega}+1+e^{-j\omega}) = \frac{1}{3}(1+2\cos\omega)
\end{aligned}
$$

Evaluating $|H(e^{j\omega})|$ for $-\pi \le \omega \le \pi$, we have

| $\omega$ (rad/s) | $\lvert H(e^{j\omega})\rvert$ |
|---|---|
| $-\pi$ | 0.3333 |
| -3 | 0.3267 |
| -2 | 0.0559 |
| -1 | 0.6935 |
| 0 | 1.0000 |
| 1 | 0.6935 |
| 2 | 0.0559 |
| 3 | 0.3267 |
| $\pi$ | 0.3333 |

The plots are shown below:-




Check answer in Matlab using the following commands

```
>> num=[1/3 1/3 1/3];
>> den=[0 1 0];
>> w=[-pi:0.01:pi];
>> [H,W]=freqz(num,den,w)
>> plot(W,abs(H))
```

**Challenge Question**:
Why does `fvtool` give an error for 3 point moving-average FIR filter coefficients defined above?