AUSTRALIAN NATIONAL UNIVERSITY
Department of Engineering

ENGN6612/4612 Digital Signal Processing and Control
Problem Set #4 Difference Equations

Q1

Find the system transfer function $H(z) = Y(z)/X(z)$ when the LTI system is described by the following difference equation:

(a) $y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$

(b) $y[n] = y[n-1] + y[n-2] + x[n-1]$

(c) $y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n-1] - x[n-2]$ (challenge problem)

Also draw the pole-zero plot for $H(z)$ and determine if the system is stable or unstable.

Q2

Consider a discrete time LTI system with following impulse response $h[n]$ and input function $x[n]$:

(a)

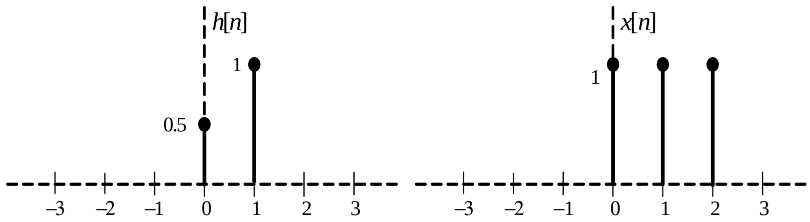

Figure 1: Figure Q2(a)

(b)

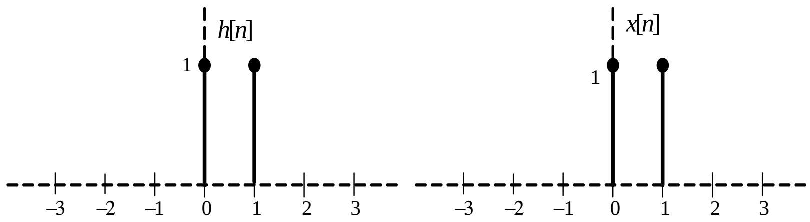

Figure 2: Figure Q2(b)

(c) $x[n] = \delta[n] - \delta[n-1]$ and
$h[n] = \delta[n] + \delta[n-1] + 0.5\delta[n-2] + 0.5\delta[n-3]$ (challenge problem).

Determine the output $y[n]$ using both (i) graphical discrete time convolution and (ii) z-transform method.

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Problem Set #4 Solution

Q1

(a) Complete Solution

The given difference equation is

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

We know

$$x[n-n_0] \quad \longleftrightarrow \quad \frac{X(z)}{z^{n_0}}$$

Using the time shift property, we have

$$\begin{aligned} y[n] \quad &\longleftrightarrow \quad Y(z) \\ y[n-1] \quad &\longleftrightarrow \quad \frac{Y(z)}{z} \\ y[n-2] \quad &\longleftrightarrow \quad \frac{Y(z)}{z^2} \end{aligned}$$

and

$$\begin{aligned} x[n] \quad &\longleftrightarrow \quad X(z) \\ x[n-1] \quad &\longleftrightarrow \quad \frac{X(z)}{z} \end{aligned}$$

Taking the z-transform of both sides of the difference equation, we have

$$Y(z) - \frac{1}{2}\frac{Y(z)}{z} \quad = \quad X(z) + \frac{1}{3}\frac{X(z)}{z}$$

Simplifying, we have

$$\begin{aligned} \left(1 - \frac{1}{2z}\right)Y(z) \quad &= \quad \left(1 + \frac{1}{3z}\right)X(z) \\ \frac{Y(z)}{X(z)} \quad &= \quad \frac{z + \frac{1}{3}}{z - \frac{1}{2}} \end{aligned}$$

Hence the transfer function in standard form is

$$H(z) \quad = \quad \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

From the transfer function, $H(z)$ has pole at $z = \frac{1}{2}$ and a zero at $z = -\frac{1}{3}$.
As the pole lies within the unit circle $|z| = 1$, system is stable.

The pole-zero map is shown in the figure below:

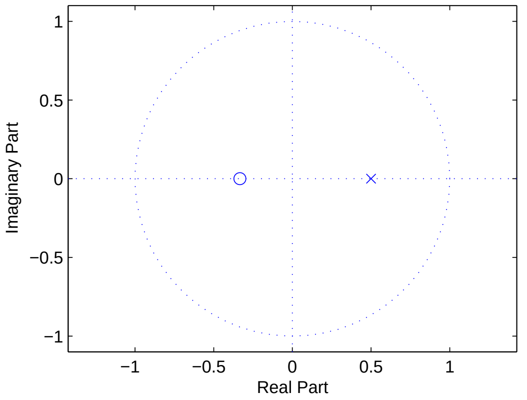


Figure 3: Question 1(a)

Additional student exercise

Take the inverse z-transform of $H(z)$ and show that the corresponding impulse response $h[n]$ is

$$h[n] \quad = \quad -\frac{2}{3}\delta[n] + \frac{5}{3}(1/2)^n u[n]$$

(Hint: use the method based on partial fractions.)

For $0 < n < 4$, the plot of $h[n]$ is shown below:-

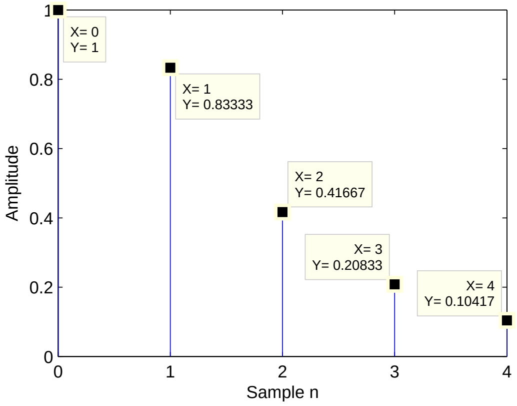


Figure 4: Question 1(a)

(b) Partial Solution

The given difference equation is

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

Taking the z-transform of both sides of the difference equation , we have

$$Y(z) = \frac{Y(z)}{z} + \frac{Y(z)}{z^2} + \frac{X(z)}{z}$$

Simplify and show that the transfer function in standard form is

$$H(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1}$$

From the transfer function, $H(z)$ has poles at $z = -0.618, 1.618$ and a zero at $z = 0$.
As one of the poles lies outside the unit circle $|z| = 1$, system is unstable.
The pole-zero map is shown in the figure below:

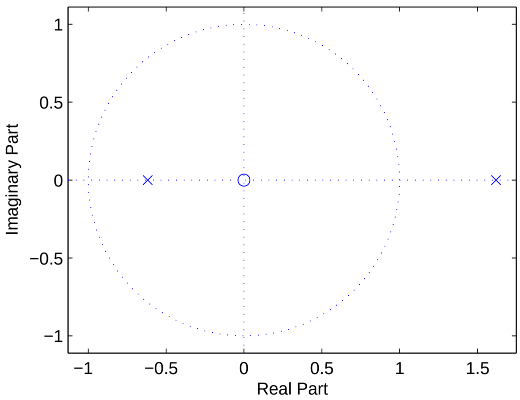


Figure 5: Question 1(b)

(c) Solution

Show that the transfer function is

$$H(z) = \frac{z(-2z^2 + \frac{13}{8}z - \frac{3}{8})}{(z+1)(z+\frac{1}{2})(z-\frac{1}{4})} = \frac{-2 + \frac{13}{8}z^{-1} - \frac{3}{8}z^{-2}}{1 + \frac{5}{4}z^{-1} + \frac{1}{8}z^{-2} - \frac{1}{8}z^{-3}}$$

System is stable (poles at $z = -1, -0.5, 0.25$ and zeros at $z = 0, 0.4063 \pm j0.15$).
The pole-zero map is shown in the figure below:

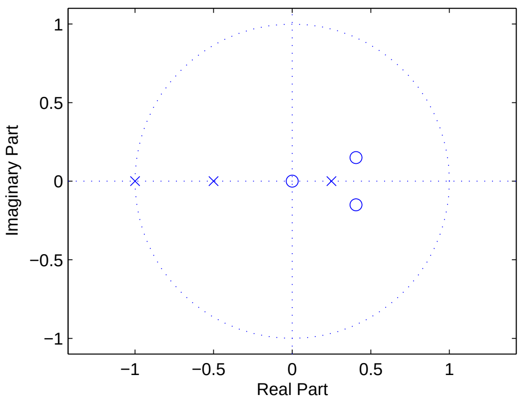


Figure 6: Question 1(c)

Q2

(a) Complete Solution

Please see pages 6-8.

(b) Solution

Solution in time domain

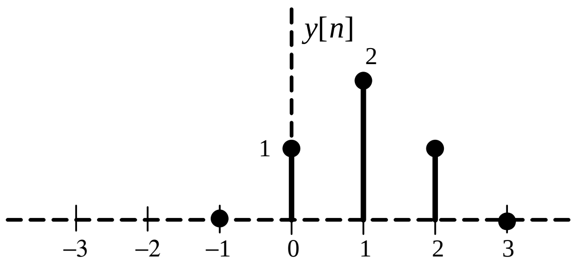


Figure 7: Question 2(b)

Solution in *z* domain

$$
\begin{aligned}
Y(z) &= \left(\frac{z+1}{z}\right)^2 = 1+\frac{2}{z}+\frac{1}{z^2} \\
y[n] &= \delta[n]+2\delta[n-1]+\delta[n-2]
\end{aligned}
$$

(c) Solution

Solution in time domain

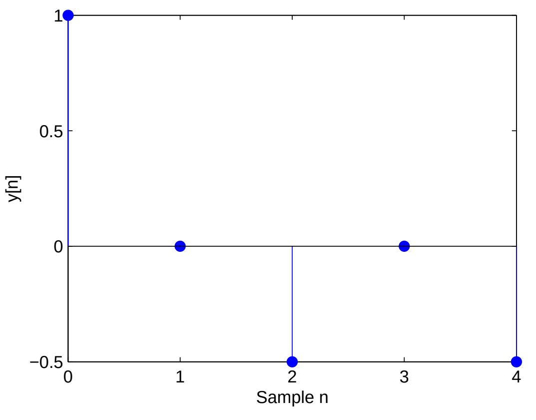


Figure 8: Question 2(c)

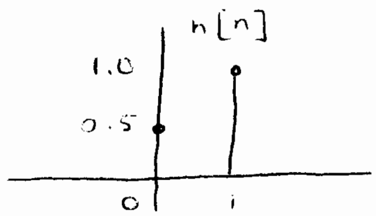
Solution in *z* domain

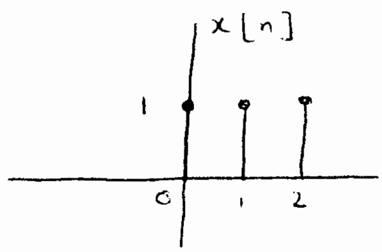
$$
\begin{aligned}
X(z) &= 1-\frac{1}{z} \\
H(z) &= 1+\frac{1}{z}+\frac{0.5}{z^2}+\frac{0.5}{z^3} \\
Y(z) &= H(z)X(z) = 1-\frac{0.5}{z^2}-\frac{0.5}{z^4} \\
y[n] &= \delta[n]-0.5\delta[n-2]-0.5\delta[n-4]
\end{aligned}
$$

GRAPHICAL DISCRETE TIME CONVOLUTION

We know $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

The given waveforms are



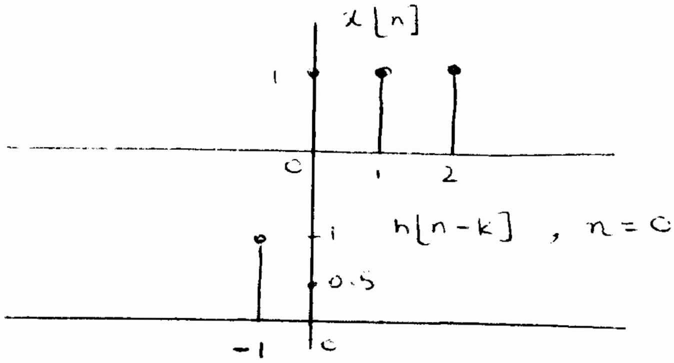


$n < 0$

When n is less than 0, there is no overlap.

Hence $y[n] = 0$

$n = 0$

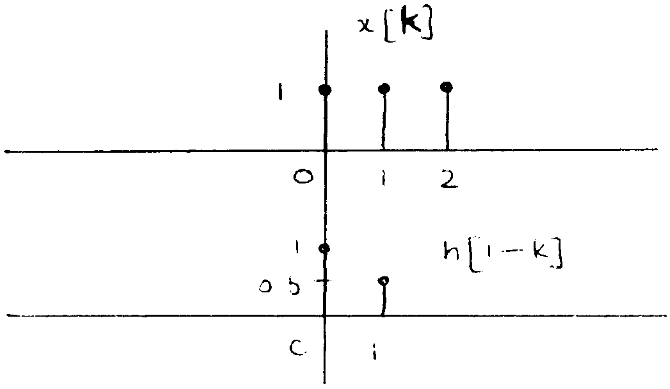


$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[0-k]$$

$$= (0.5)(1)$$

$$= 0.5$$

n = 1

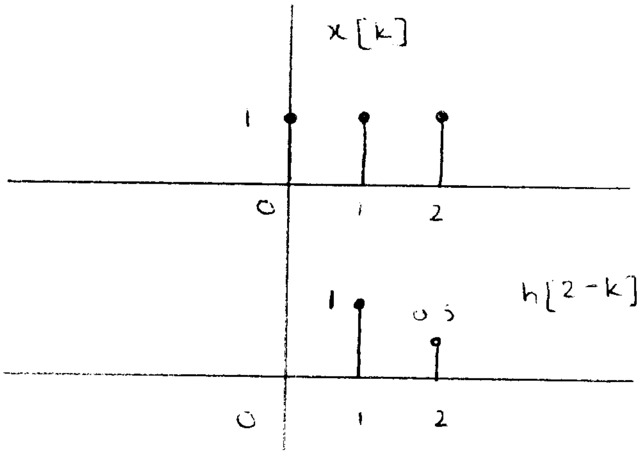


$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\,h[1-k]$$

$$= (1)(1) + (1)(0.5) + (2)(0)$$

$$= 1 + 0.5 = 1.5$$

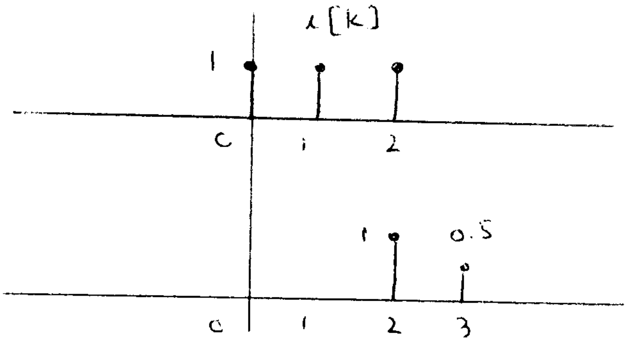
n = 2



$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\,h[2-k]$$

$$= (1)(0) + (1)(1) + (1)(0.5) = 1.5$$

n = 3

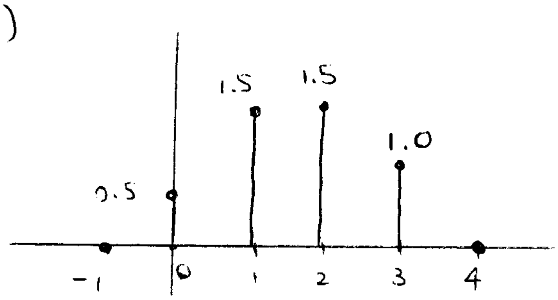


$$y[3] = \sum_{k=-\infty}^{\infty} x[k]\,h[3-k]$$

$$= (1)(0) + (1)(0) + (1)(1) + (0)(0.5)$$

$$= 1$$

n = 4 There is no overlap
Hence $y[n] = 0$ for $n > 4$.
Plot of $y[n]$ is shown opposite



SOLUTION IN Z-DOMAIN

The equation for impulse response $h[n]$ is

$h[n] = 0.5\delta[n] + \delta[n-1]$

The equation for input $x[n]$ is

$x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

Taking the z-transform, we have

$H(z) = 0.5 + \frac{1}{z}$ $\left(\because \delta[n] \longleftrightarrow 1 \quad \delta[n-k] \longleftrightarrow \frac{1}{z^k} \right)$

$X(z) = 1 + \frac{1}{z} + \frac{1}{z^2}$

For a discrete-time LTI system,

$$Y(z) = H(z)\,X(z)$$

$$= \left[0.5 + \frac{1}{z}\right]\left[1 + \frac{1}{z} + \frac{1}{z^2}\right]$$

$$= 0.5 + \frac{0.5}{z} + \frac{0.5}{z^2} + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$$

$$= 0.5 + \frac{1.5}{z} + \frac{1.5}{z^2} + \frac{1}{z^3}$$

Taking the inverse z-transform,

$$y[n] = 0.5\delta[n] + 1.5\delta[n-1] + 1.5\delta[n-2] + \delta[n-3]$$

Check :- This is the same output as found using discrete-time convolution