AUSTRALIAN NATIONAL UNIVERSITY
Department of Engineering

ENGN6612/4612 Digital Signal Processing and Control
Problem Set #3 Inverse z-Transform

Q1

Using properties of z-transform, find the z-transform of the following discrete-time functions:
(a) $n\,u[n]$
(b) $n^2\,u[n]$
(c) $nc^n u[n]$ `(challenge problem)`
(d) $u[k-2]$

Q2

Using the method based on partial fraction expansion, find x[n] if X(z) equals:

(a) $\dfrac{z+1}{(z-2)(z+3)}$
(b) $\dfrac{2z-3}{z(z-0.5)(z+0.3)}$
(c) $\dfrac{z}{(z-1)(z-4)}$
(d) $\dfrac{100z^2}{(z-1.1)(z-1)}$
(e) $\dfrac{0.1z(z+1)}{(z-1)^2(z-0.6)}$ `(challenge problem)`

Also plot $x[n]$ for $0 < n < 4$

AUSTRALIAN NATIONAL UNIVERSITY
Department of Engineering

ENGN6612/4612 Digital Signal Processing and Control
Problem Set #3 Solution

Q1

(a) Complete Solution

We know

$$u[n] \longleftrightarrow \frac{z}{z-1}$$

Hence

$$\begin{aligned}\mathcal{Z}\{nu[n]\} &= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) \\ &= -z\frac{(z-1)(1)-z(1)}{(z-1)^2} \\ &= \frac{z}{(z-1)^2}\end{aligned}$$

(b) Solution with Hint

Show that

$$n^2u[n] \longleftrightarrow \frac{z(z+1)}{(z-1)^3}$$

Hint:-

$$n\,u[n] \longleftrightarrow \frac{z}{(z-1)^2}$$

(c) Solution with Hint

Show that

$$nc^n\,u[n] \longleftrightarrow \frac{zc}{(z-c)^2}$$

Hint:-

$$c^n\,u[n] \longleftrightarrow \frac{z}{z-c}$$

(d) Complete Solution

We know

$$u[n] \longleftrightarrow \frac{z}{z-1}$$

and

$$\mathcal{Z}\{x[n-n_0]\} = \frac{X(z)}{z^{n_0}}$$

Hence using the time shifting property,

$$\begin{aligned}\mathcal{Z}\{u[n-2]\} &= \frac{\frac{z}{z-1}}{z^2} \\ &= \frac{1}{z(z-1)}\end{aligned}$$

Q2

(a) Complete Solution

Given that

$$X(z) = \frac{z+1}{(z-2)(z+3)}$$

Rewriting

$$X(z) = \frac{z(z+1)}{z(z-2)(z+3)} = z\left[\frac{(z+1)}{z(z-2)(z+3)}\right]$$

Using partial fraction expansion, we have

$$\frac{(z+1)}{z(z-2)(z+3)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+3}$$

Evaluating the coefficients,

$$\begin{aligned} A &= \lim_{z\to 0}\left[\frac{z+1}{(z-2)(z+3)}\right] = -\frac{1}{6} \\ B &= \lim_{z\to 2}\left[\frac{z+1}{(z)(z+3)}\right] = \frac{3}{10} \\ C &= \lim_{z\to -3}\left[\frac{z+1}{(z)(z-2)}\right] = -\frac{2}{15} \end{aligned}$$

Hence,

$$\begin{aligned} X(z) &= z\left[\frac{-\frac{1}{6}}{z} + \frac{3}{10}\frac{1}{z-2} - \frac{2}{15}\frac{1}{z+3}\right] \\ &= -\frac{1}{6} + \frac{3}{10}\frac{z}{z-2} - \frac{2}{15}\frac{z}{z+3} \end{aligned}$$

Taking the inverse z-transform, we have

$$x[n] = -\frac{1}{6}\delta[n] + \frac{3}{10}(2)^n u[n] - \frac{2}{15}(-3)^n u[n]$$

For $0 < n < 4$, we have

$$\begin{aligned} x[0] &= -\frac{1}{6} + (0.3)(2)^0 - \frac{2}{15}(-3)^0 = 0 \\ x[1] &= 0 + (0.3)(2)^1 - \frac{2}{15}(-3)^1 = 1 \\ x[2] &= 0 + (0.3)(2)^2 - \frac{2}{15}(-3)^2 = 0 \\ x[3] &= 0 + (0.3)(2)^3 - \frac{2}{15}(-3)^3 = 6 \\ x[4] &= 0 + (0.3)(2)^4 - \frac{2}{15}(-3)^4 = -6 \end{aligned}$$

Using the initial value theorem to check the value of $x[0]$, we have

$$\begin{aligned} x[0] &= \lim_{z\to\infty} X(z) \\ &= \lim_{z\to\infty}\frac{z+1}{(z-2)(z+3)} = \lim_{z\to\infty}\frac{z+1}{z^2+z-6} = \lim_{z\to\infty}\frac{z^{-1}+z^{-2}}{1+z^{-1}-6z^{-2}} = \frac{0+0}{1+0+0} = 0 \end{aligned}$$

The plot of $x[n]$ is shown below:-

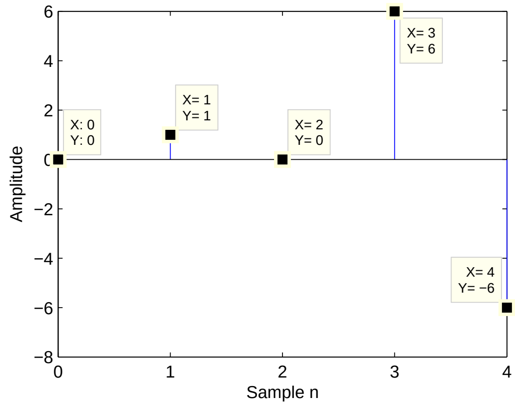

Figure 1: Question 1(a)

(b) Partial Solution

Given that

$$X(z) = \frac{2z-3}{z(z-0.5)(z+0.3)}$$

Rewriting

$$\begin{aligned} X(z) &= \frac{z(2z-3)}{z^2(z-0.5)(z+0.3)} \\ &= z\left[\frac{(2z-3)}{z^2(z-0.5)(z+0.3)}\right] \end{aligned}$$

Using partial fraction expansion, we have

$$\frac{(2z-3)}{z^2(z-0.5)(z+0.3)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-0.5} + \frac{C}{z+0.3}$$

Evaluating the coefficients,

$$\begin{aligned} B &= \lim_{z\to 0}\left[\frac{(2z-3)}{(z-0.5)(z+0.3)}\right] = 20 \\ A &= \lim_{z\to 0}\frac{d}{dz}\left[\frac{(2z-3)}{(z-0.5)(z+0.3)}\right] = -40 \\ C &= \lim_{z\to 0.5}\left[\frac{(2z-3)}{(z^2)(z+0.3)}\right] = -10 \\ D &= \lim_{z\to -0.3}\left[\frac{(2z-3)}{(z^2)(z-0.5)}\right] = 50 \end{aligned}$$

Hence,

$$\begin{aligned} X(z) &= z\left[-\frac{40}{z} + \frac{20}{z^2} - \frac{10}{z-0.5} + \frac{50}{z+0.3}\right] \\ &= -40 + \frac{20}{z} - 10\frac{z}{z-0.5} + 50\frac{z}{z+0.3} \end{aligned}$$

Taking the inverse z-transform, we have

$$x[n] = -40\delta[n] + 20\delta[n-1] - 10(0.5)^n u[n] + 50(-0.3)^n u[n]$$

The plotting is left as an exercise for the students.
Check your answer via Matlab or by comparing your answer with another student.

(c) Solution

Show that

$$x[n] = \frac{1}{3}(4^n - 1)u[n]$$

The plot of $x[n]$ is shown below:-

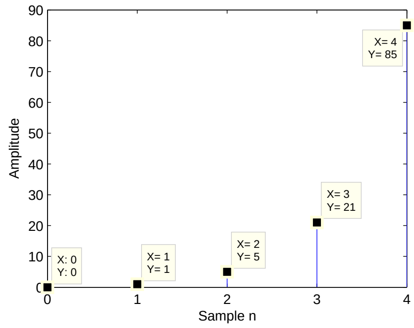

Figure 2: Question 1(c)

(d) Solution

Show that

$$x[n] = 1100(1.1)^n u[n] - 1000u[n]$$

The plot of $x[n]$ is shown below:-

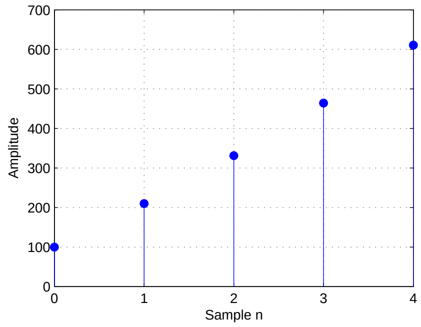

Figure 3: Question 1(d)

(e) Solution

Show that

$$x[n] = \{0.5n - 1 + (0.6)^n\}u[n]$$

The plot of $x[n]$ is shown below:-

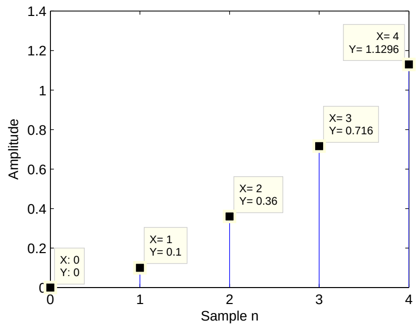

Figure 4: Question 1(e)