AUSTRALIAN NATIONAL UNIVERSITY
Department of Engineering

ENGN6612/4612 Digital Signal Processing and Control
Problem Set #2 z-Transform

Q1

Using the definition of the z-transform, find the z-transform of the following discrete-time functions:
(a) $\delta[n]$
(b) $\delta[n-k]$
(c) $u[n]$
(d) $c^n\, u[n]$ where c is a complex constant
(e) $\sin[\omega n]u[n]$
(f) $\cos[\omega n]u[n]$
(g) $r^n \sin[\omega n]u[n]$
(h) $r^n \cos[\omega n]u[n]$ (challenge problem)

Q2

Find the z-transform of the following discrete-time functions:
(a) $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$
(b) $x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$
(c) $x[n] = \left\{\frac{5}{12} + \frac{1}{3}(-2)^n - \frac{3}{4}(-3)^n\right\} u[n]$ (challenge problem)

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Problem Set #2 Solution

Q1

(a) Complete Solution

The given function is a discrete-time unit impulse given by

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Using definition of z-transform, we have

$$\begin{aligned} \mathcal{Z}\{\delta[n]\} &= \sum_{n=-\infty}^{n=+\infty} \delta[n] z^{-n} \\ &= \ldots + \delta[-2]z^{2} + \delta[-1]z^{1} + \delta[0]z^{0} + \delta[1]z^{-1} + \delta[2]z^{-2} + \ldots \\ &= \ldots + 0 + (1)(z^{0}) + 0 + \ldots \\ &= 1 \end{aligned}$$

Hence,

$$\delta[n] \quad \longleftrightarrow \quad 1$$

(b) Complete Solution

The given function is a time shifted discrete-time unit impulse given by

$$\delta[n-k] = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Using definition of z-transform, we have

$$\begin{aligned} \mathcal{Z}\{\delta[n-k]\} &= \sum_{n=-\infty}^{n=+\infty} \delta[n-k] z^{-n} \\ &= \ldots + 0 + (1)(z^{-k}) + 0 + \ldots \\ &= \frac{1}{z^{k}} \end{aligned}$$

Hence,

$$\delta[n] \quad \longleftrightarrow \quad \frac{1}{z^{k}}$$

(c) Complete Solution

The given function is a discrete-time unit step given by

$$u[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$$

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{u[n]\} &= \sum_{n=-\infty}^{n=+\infty} u[n] z^{-n} \\
&= u[0]z^0 + u[1]z^{-1} + u[2]z^{-2} + u[3]z^{-3} + \dots \\
&= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \\
&= \frac{1}{1 - \frac{1}{z}} \\
&= \frac{z}{z-1}
\end{aligned}$$

Hence,

$$u[n] \quad \longleftrightarrow \quad \frac{z}{z-1}$$

(d) Complete Solution

The given function is a discrete-time exponential.

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{c^n u[n]\} &= \sum_{n=-\infty}^{n=+\infty} c^n u[n] z^{-n} \\
&= \sum_{n=-\infty}^{n=+\infty} u[n] \left(\frac{z}{c}\right)^{-n} \\
&= u[0]\left(\frac{z}{c}\right)^{0} + u[1]\left(\frac{z}{c}\right)^{-1} + u[2]\left(\frac{z}{c}\right)^{-2} + u[3]\left(\frac{z}{c}\right)^{-3} + \dots \\
&= 1 + \left(\frac{z}{c}\right)^{-1} + \left(\frac{z}{c}\right)^{-2} + \left(\frac{z}{c}\right)^{-3} + \dots \\
&= \frac{1}{1 - \left(\frac{z}{c}\right)^{-1}} \\
&= \frac{z}{z-c}
\end{aligned}$$

Hence,

$$c^n u[n] \quad \longleftrightarrow \quad \frac{z}{z-c}$$

(e) Complete Solution

The given function is a discrete-time sine wave.

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{\sin[\omega n]u[n]\} &= \sum_{n=-\infty}^{n=+\infty} \sin[\omega n]u[n]z^{-n} \\
&= \sum_{n=-\infty}^{n=+\infty} \left(\frac{e^{j\omega n}-e^{-j\omega n}}{2j}\right) u[n]z^{-n}
\end{aligned}$$

Let

$$\begin{aligned}
c_1 &= e^{j\omega} \\
c_2 &= e^{-j\omega}
\end{aligned}$$

Hence we have,

$$\begin{aligned}
\mathcal{Z}\{\sin[\omega n]u[n]\} &= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} (c_1^n - c_2^n)\, u[n]z^{-n} \\
&= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} u[n]\left(\frac{z}{c_1}\right)^{-n} - \frac{1}{2j}\sum_{n=-\infty}^{n=-\infty} u[n]\left(\frac{z}{c_2}\right)^{-n} \\
&= \frac{1}{2j}\frac{z}{z-c_1} - \frac{1}{2j}\frac{z}{z-c_2} \qquad \text{(using result of Q1 part d)} \\
&= \frac{z(c_1-c_2)}{2j(z-c_1)(z-c_2)} \\
&= \frac{z\left(\frac{c_1-c_2}{2j}\right)}{z^2-2z\left(\frac{c_1+c_2}{2}\right)+1} \qquad (\because c_1c_2=1) \\
&= \frac{(\sin\omega)z}{z^2-(2\cos\omega)z+1}
\end{aligned}$$

Hence,

$$\sin[\omega n]u[n] \longleftrightarrow \frac{(\sin\omega)z}{z^2-(2\cos\omega)z+1}$$

(f) Solution with Hint

Show that the final answer is

$$\cos[\omega n]u[n] \longleftrightarrow \frac{z^2-(\cos(\omega))z}{z^2-(2\cos\omega)z+1}$$

Hint:-
$\cos[\omega n] = \left(\frac{e^{j\omega n}+e^{-j\omega n}}{2}\right)$

(g) Partial Solution

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{r^n \sin[\omega n]u[n]\} &= \sum_{n=-\infty}^{n=+\infty} r^n \sin[\omega n]u[n]z^{-n} \\
&= \sum_{n=-\infty}^{n=+\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) u[n] \left(\frac{z}{r}\right)^{-n}
\end{aligned}$$

Let

$$\begin{aligned}
c_1 &= e^{j\omega} \\
c_2 &= e^{-j\omega}
\end{aligned}$$

Hence we have,

$$\begin{aligned}
\mathcal{Z}\{r^n \sin[\omega n]u[n]\} &= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} (c_1^n - c_2^n)\, u[n] \left(\frac{z}{r}\right)^{-n} \\
&= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} u[n]\left(\frac{z}{rc_1}\right)^{-n} - \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} u[n]\left(\frac{z}{rc_2}\right)^{-n} \\
&= \frac{1}{2j}\frac{z}{z-rc_1} - \frac{1}{2j}\frac{z}{z-rc_2}
\end{aligned}$$

Show that this can be written in the form

$$\begin{aligned}
\mathcal{Z}\{r^n \sin[\omega n]u[n]\} &= \frac{zr\left(\frac{c_1-c_2}{2j}\right)}{z^2 - 2rz\left(\frac{c_1+c_2}{2}\right) + r^2} \\
&= \frac{(r\sin\omega)z}{z^2 - (2r\cos\omega)z + r^2}
\end{aligned}$$

Hence,

$$r^n \sin[\omega n]u[n] \quad \longleftrightarrow \quad \frac{(r\sin\omega)z}{z^2 - (2r\cos\omega)z + r^2}$$

(h) Solution

Show that the final answer is

$$r^n \cos[\omega n]u[n] \quad \longleftrightarrow \quad \frac{z^2 - (r\cos(\omega))z}{z^2 - (2r\cos\omega)z + r^2}$$

Check answer in Matlab using the following commands

```
>> syms w n z
>> f=(r^n)*cos(w*n)*heaviside(n);
>> Ans=maple('ztrans',f,n,z)
>> pretty(Ans)
```

Q2

(a) Complete Solution

Given that,

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

We know that

$$c^n u[n] \longleftrightarrow \frac{z}{z-c}$$

Hence,

$$\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{3}}$$

$$7\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{7z}{z-\frac{1}{3}}$$

Also,

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{2}}$$

$$6\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{6z}{z-\frac{1}{2}}$$

The z-transform of the given function $X(z)$ is thus given by

$$\begin{aligned} X(z) &= \frac{7z}{z-\frac{1}{3}} - \frac{6z}{z-\frac{1}{2}} \\ &= \frac{7z(z-\frac{1}{2}) - 6z(z-\frac{1}{3})}{(z-\frac{1}{3})(z-\frac{1}{2})} \\ &= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})} \end{aligned}$$

(b) Solution

$$X(z) = \frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

(c) Solution

$$X(z) = \frac{z(2z+3)}{(z+2)(z+3)(z-1)}$$

Check answer in Matlab using the following commands

```
>> syms n z
>> f= ((5/12) + ((1/3)*(-2)^n) -((3/4)*(-3)^n))*heaviside(n);
>> Ans=maple('ztrans',f,n,z)
>> pretty(simplify(Ans))
```