

## 1 Aim

1. To revise and/or acquire and develop skills in using MATLAB simulations.
2. To develop an understanding of state space concepts.
3. To learn the use of the LQR.

## 2 Controller Design

This section concerns controller design for the inverted pendulum studied in lectures:

$$\begin{pmatrix} x_{n+1}^1 \\ x_{n+1}^2 \end{pmatrix} = \begin{pmatrix} 1 & T_s \\ T_s g & 1 \end{pmatrix} \begin{pmatrix} x_n^1 \\ x_n^2 \end{pmatrix} + \begin{pmatrix} 0 \\ T_s/m \end{pmatrix} u_n \quad (1)$$

$$y_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_n^1 \\ x_n^2 \end{pmatrix} \quad (2)$$

Use the following parameters:

$$g = 10, \quad m = 0.01, \quad T_s = 0.01.$$

QUESTION 2.1    1. *Is the state space realisation (6) reachable?*

2. *Find a stabilising state feedback control law  $u = kx$ .*
3. *Is the state space realisation (6), (2) observable?*
4. *Design an asymptotic observer with stable error dynamics.*
5. *Combine the state feedback control law with the observer to obtain an observer-controller, and check stability.*

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### MATLAB EXERCISE 2.2

pend-state-fb.mdl, pend-output-fb.mdl

1. Download the MATLAB SIMULINK files and save in your working directory.
2. State feedback control.

(a) Open pend-state-fb.mdl. The pendulum state space block has been set up using the above model parameters, but with  $y = x$  (a vector) for state feedback. Double click on the pendulum block and inspect the parameters. The simulation has been set up for a time duration 1 second - to see, this look at Simulation -> Simulation parameters... The controller gain block has been set up with zero values for an initial open loop simulation - double click to inspect. The input block has been set up to provide a unit impulse signal.

- (b) Run the simulation (Simulation  $\rightarrow$  Start, or click the start button), and note the impulse response. Is the system open loop stable? Why?
- (c) Enter your state feedback controller gains, and simulate. Note the impulse response. Is the system closed loop stable? Why?

3. Output feedback control.

- (a) Open pend-output-fb.mdl. The pendulum state space block has been set up using the above model parameters, with  $y = x^1$  (a scalar) for output feedback. Double click on the pendulum block and inspect the parameters. The simulation has been set up for a time duration 1 second. The controller gain block has been set up with zero values for an initial open loop simulation. The input block has been set up to provide a unit impulse signal. The observer state space block has been set up with dummy values.
- (b) Enter your state feedback controller gains, and your observer design (you will need to enter values for the observer's  $A$  and  $B$  matrices; keep  $C = I$  and  $D = 0$ ). Simulate. Note the impulse response. Is the system closed loop stable? Why?
- (c) Examine the state estimate trajectories. What can you say about the error?

### 3 Reachability and Digital Filter Representation

This question concerns the digital filter shown in Figure 1.

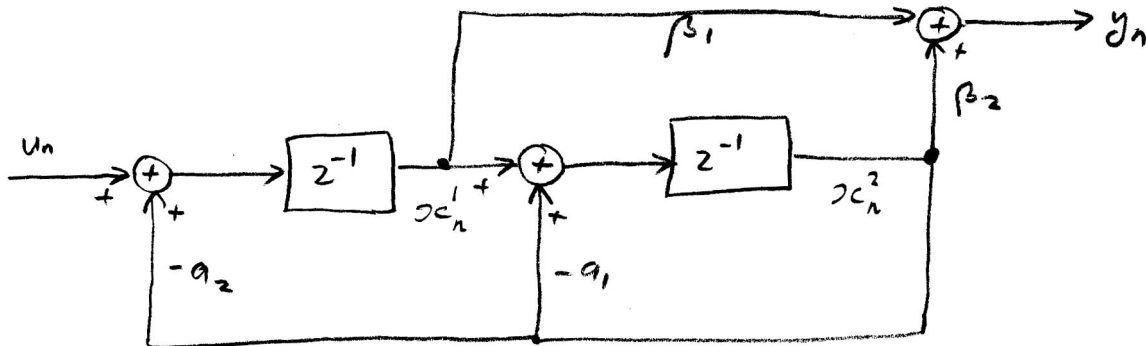


Figure 1: Reachable digital filter.

QUESTION 3.1 1. By inspection of the digital filter of Figure 1, apply a unit impulse  $u_n = \delta_n$  and hence show that  $\beta_1 = h_1$ ,  $\beta_2 = h_2$ , where  $h_n$  is the impulse response. You will need to assume that the initial state is zero:  $x_0^1 = 0$ ,  $x_0^2 = 0$ .

- Again by inspection, find an input  $u_0, u_1$  that takes the zero initial state  $x_0 = (0, 0)$  to a desired final state  $x_2 = (a, b)$ .
- Find the matrices  $A, B$  and  $C$  in the state space equations for the digital filter of the form

$$\begin{aligned} x_{n+1} &= Ax_n + Bu_n \\ y_n &= Cx_n \end{aligned} \quad (3)$$

- Is the system (3) you found for the digital filter of Figure 1 reachable? Is this consistent with your result of part (ii)?
- Verify the expression for the impulse response

$$h_1 = CB, \quad h_2 = CAB$$

- Find the transfer function

$$\mathcal{H}(z) = C(zI - A)^{-1}B$$

## 4 Observability and Digital Filter Representation

This question concerns the digital filter shown in Figure 2.

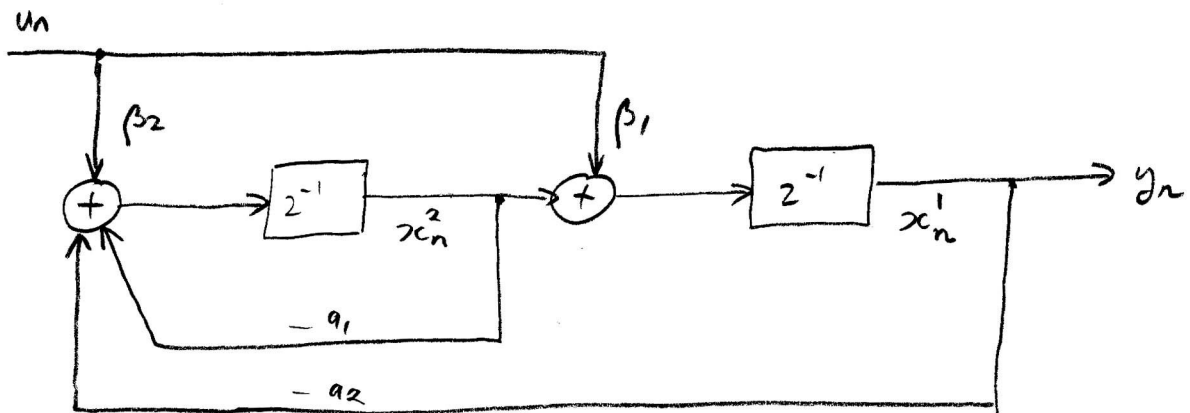


Figure 2: Observable digital filter.

- QUESTION 4.1
- By inspection of the digital filter of Figure 2, apply a unit impulse  $u_n = \delta_n$  and hence show that  $\beta_1 = h_1, \beta_2 = h_2$ , where  $h_n$  is the impulse response. You will need to assume that the initial state is zero:  $x_0^1 = 0, x_0^2 = 0$ .
  - Assume the initial state is  $x_0 = (a, b)$ , and the input signal is zero ( $u_n = 0$ ). By inspection, show how the initial state can be found from the measurements  $y_0, y_1$ . Repeat for non-zero input.
  - Find the matrices  $A, B$  and  $C$  in the state space equations for the digital filter of the form (3).

4. Is the system (3) you found for the digital filter of Figure 2 observable? Is this consistent with your result of part (ii)?
5. Verify the expression for the impulse response

$$h_1 = CB, \quad h_2 = CAB$$

6. Find the transfer function

$$\mathcal{H}(z) = C(zI - A)^{-1}B$$

## 5 Reachability

This question concerns the linear system

$$x_{n+1} = Ax_n + Bu_n$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- QUESTION 5.1
1. Find the reachability matrix  $\mathcal{C}(A, B)$  and determine whether or not the system is reachable.
  2. Let the initial state be zero:  $\xi_0 = 0$ . Find the reachable subspace consisting of states  $\xi_f$  that can be reached from zero. Sketch the subspace in the 2-d state space.
  3. If  $\xi_f = (5, 0)$ , determine an input  $u_0, u_1$  that takes the system from zero at time  $n = 0$  to  $(5, 0)$  at time  $n = 2$  (two steps). Can this state be reached in 1 step? If so, how?
  4. Show that all initial states  $\xi_0 = (a, b)$  can be controlled to the zero state  $\xi_f = (0, 0)$  in at most two steps.

## 6 Observability

This question concerns the linear system

$$\begin{aligned} x_{n+1} &= Ax_n \\ y_n &= Cx_n \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = (1 \ 1)$$

- QUESTION 6.1
1. Find the observability matrix  $\mathcal{O}(C, A)$  and determine whether or not the system is observable.
  2. Show that it is only possible to observe the sum  $x_0^1 + x_0^2$ .

## 7 LQR

In this section we will do some practice with LQR design.

First, we consider the one dimensional linear system

$$x_{n+1} = 2x_n + 1u_n \quad (4)$$

and cost function

$$J = \sum_{n=0}^1 [x_n^2 + u_n^2] + x_2^2 \quad (5)$$

(Here, since everything is one dimensional,  $x_i^2 = x_i \times x_i$ , etc.)

QUESTION 7.1 1. Find values  $X_2$ ,  $X_1$  and  $X_0$  such that the value function is given by

$$V(x_n, n) = X_n x_n^2$$

( $n = 0, 1, 2$ ) and find the corresponding state feedback gains

$$u_0 = -K_0 x_0, \quad u_1 = -K_1 x_1$$

2. Let the initial state be  $x_0 = 5$ . Find:

- (a) The optimal control signal  $u_0, u_1$  and the optimal state trajectory  $x_0, x_1, x_2$ .
- (b) The optimal cost.

In the following MATLAB exercise, we will use LQR to find stabilizing state feedback gains. The state space equations are

$$\begin{pmatrix} x_{n+1}^1 \\ x_{n+1}^2 \end{pmatrix} = \begin{pmatrix} 1 & T_s \\ T_s g & 1 \end{pmatrix} \begin{pmatrix} x_n^1 \\ x_n^2 \end{pmatrix} + \begin{pmatrix} 0 \\ T_s/m \end{pmatrix} u_n \quad (6)$$

Use the following parameters:

$$g = 10, \quad m = 0.01, \quad T_s = 0.01.$$

Select your own values of  $Q$  and  $R$  for LQR design.

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MATLAB EXERCISE 7.2

pend-state-fb.mdl, pend-state-fb-step.mdl

1. Using MATLAB's help system, read about the `dlqr` command.
2. Use the command

$$[K, X, E] = \text{dlqr}(A, B, Q, R)$$

to determine the feedback gains, the solution of the ARE, and the closed loop eigenvalues.

3. Check directly the eigenvalues of the closed loop matrix

$$A - BK$$

4. Use the SIMULINK model `pend-state-fb.mdl` from CLAB3 to simulate the closed loop system and observe the unit impulse response with the optimal state feedback gains.
  5. Now use `pend-state-fb-step.mdl` to observe the step response.
  6. Experiment with different choices of  $Q$  and  $R$  and examine the effect on speed of response and control actions.
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## 8 Miscellaneous

QUESTION 8.1 1. Show that  $(A, B)$  reachable is equivalent to  $(B', A')$  observable.

2. Show that  $(C, A)$  observable is equivalent to  $(A', C')$  reachable.

3. Let  $T$  be an invertible  $N_x \times N_x$  matrix. Let  $(A_1, B_1, C_1)$  specify a state space linear system with transfer function  $\mathcal{H}_1(z)$ . Let  $(A_2, B_2, C_2)$  specify a second state space linear system where

$$A_2 = T^{-1}A_1T, \quad B_2 = T^{-1}B_1, \quad C_2 = C_1T$$

Find the transfer function  $\mathcal{H}_2(z)$  of this second system and hence show that  $\mathcal{H}_1(z) = \mathcal{H}_2(z)$ . Explain the significance of this result.

4. The Fibonacci sequence  $\{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$  is generated by the equation

$$\begin{aligned} y_n &= y_{n-1} + y_{n-2}, \quad k \geq 2, \\ y_0 &= 0, \quad y_1 = 1 \end{aligned}$$

(a) Show that

$$y_n = \frac{1}{\sqrt{5}}(\lambda_+^n - \lambda_-^n)$$

where

$$\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

(b) Show that

$$\lim_{n \rightarrow \infty} \frac{\log y_n}{n} = \frac{1 + \sqrt{5}}{2}$$

5. Consider the RLC circuit shown in Figure 3.

Here,  $v_g$  is a voltage source,  $i$  is the current through all devices (it's a series circuit), and  $v$  is the capacitor voltage. The equations for this circuit are:

$$v_L = L \frac{di}{dt}, \quad i = C \frac{dv}{dt}$$

and

$$v_g = iR + L \frac{di}{dt} + v$$

Define the state vector

$$x(t) = \begin{pmatrix} i(t) \\ v(t) \end{pmatrix}$$

and find the (continuous time) state equations. Use  $y = v$  as the output.

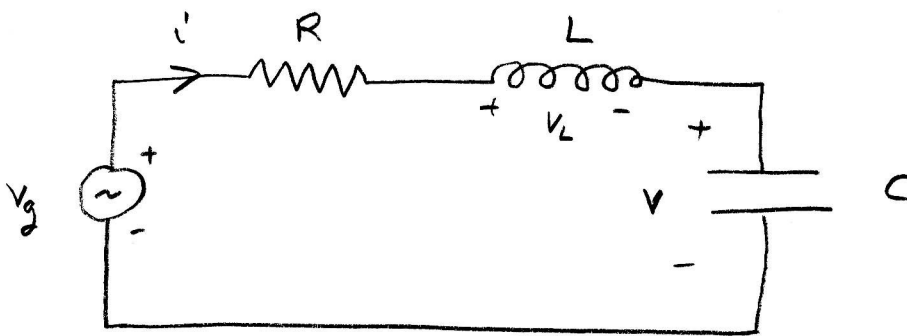


Figure 3: RLC circuit.