Q1 (1)

(a)

$$x_1^2 + x_1 x_2 + \frac{x_2^2}{2} = \frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)$$

expand both sides.

$$RHS = \frac{x_1^2}{2(1-\rho^2)\sigma_1^2} + x_1\left(\frac{-2\mu_1}{2(1-\rho^2)\sigma_1^2} + \frac{2\rho\mu_2}{2(1-\rho^2)\sigma_1\sigma_2}\right)$$

$$\frac{x_2^2}{2(1-\rho^2)\sigma_2^2} + x_2\left(\frac{-2\mu_2}{2(1-\rho^2)\sigma_2^2} + \frac{2\rho\mu_1}{2(1-\rho^2)\sigma_1\sigma_2}\right)$$ (1)

$$-\frac{2\rho x_1 x_2}{2(1-\rho^2)\sigma_1\sigma_2} + \underbrace{\left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{2\rho\mu_1\mu_2}{\sigma_1\sigma_2}\right)\frac{1}{2(1-\rho^2)}}$$

$$\frac{\cancel{x_1^2}}{2(1-\rho^2)\sigma_1^2} = \cancel{x_1^2}$$

$$\frac{\cancel{x_2^2}}{2(1-\rho^2)\sigma_2^2} = \frac{\cancel{x_2^2}}{2}$$ (1)

$$\frac{-\cancel{2}\rho}{\cancel{2}(1-\rho^2)\sigma_1\sigma_2}\cancel{x_1x_2} = \cancel{x_1x_2}$$

$$\left(\frac{\mu_2\rho}{\sigma_1\sigma_2} - \frac{\mu_1}{\sigma_1^2}\right)\cancel{\frac{1}{(1-\rho^2)}} = 0$$

$$\left(\frac{\rho\mu_1}{\sigma_1\sigma_2} - \frac{\mu_2}{\sigma_2^2}\right)\cancel{\frac{1}{(1-\rho^2)}} = 0$$

$$\left(\frac{\mu_1^2}{\sigma_1^2} - \frac{2\rho\mu_1\mu_2}{\sigma_1\sigma_2} + \frac{\mu_2^2}{\sigma_2^2}\right)\cancel{\frac{1}{2(1-\rho^2)}} = 0$$

$\Uparrow$

$\mu_1 = \mu_2 = 0$ solves eq^ns on right!

Ⓐ $\sigma_1^2 = \frac{1}{2(1-\rho^2)}$

Ⓑ $\sigma_2^2 = \frac{1}{(1-\rho^2)}$

$\frac{1}{2}$

Ⓒ $\sigma_1\sigma_2 = \frac{-\rho}{(1-\rho^2)}$

$$Ⓒ = \frac{-\rho}{(1-\rho^2)} = \frac{1}{\sqrt{2(1-\rho^2)}} \frac{1}{\sqrt{(1-\rho^2)}} = \frac{1}{(1-\rho^2)} \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \rho = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sigma_1^2 = \frac{1}{2(1-\frac{1}{2})} = 1$$

$$\sigma_2^2 = \frac{1}{(1-\frac{1}{2})} = 2$$

$\frac{1}{2}$

$$\sigma_1 = 1, \quad \sigma_2 = \sqrt{2}, \quad \rho = -\frac{1}{\sqrt{2}}$$

b) i) Marginal distribution.

$$\phi(x_1) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right)$$

1

1.3

b ii) Correlation $E[x_1 x_2] = \rho = -\frac{1}{\sqrt{2}}$ 1

c) Note that

$$x_1^2 + x_1 x_2 + \frac{x_2^2}{2} = (x_1 \quad x_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= a x_1^2 + 2b x_1 x_2 + c x_2^2$$

$a = 1 \quad b = 1/2 \quad c = \frac{1}{2}$

Hence,

$$\frac{1}{2}\Sigma^{-2} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$ 1/2

$$\Sigma^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

~~Also note~~ $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 1/2

Also note $\Sigma^2 = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}}\sqrt{2} \\ -\frac{1}{\sqrt{2}}\sqrt{2} & (\sqrt{2})^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$\det(\Sigma^2) = 2 - 1 = 1$

$\phi(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\xi^T \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\xi\right)$ 1

Q2

a i) Sample space is all sets of possible rolls.

$S = \{(1,1),(1,2),\dots(1,6),(2,1),\dots,(2,6),\dots,(6,1),\dots(6,5),(6,6)\}$ 1

ii) The prob dist. $\phi : S \to \mathbb{R}$.

$\phi_S(i,j) = \phi(i)\phi(j)$ $i,j=1,\dots 6$, where $\phi(i), \phi(j)$ are given by the table. 1

b) i) $\{A \geq 4\} = \{(i,j) \in S \mid j \in \{4,5,6\}, i \in \{1,2,3,\dots,6\}\}$

$$P(A \geq 4) = \sum_{j=1}^{6} \sum_{j=4}^{6} \phi_S(i,j)$$

$$= \sum_{i=4}^{6} \sum_{j=1}^{6} \phi(i)\phi(j)$$

$$= \sum_{i=4}^{6} \phi(i) \underbrace{\sum_{j=1}^{6} \phi(j)}_{1}$$

$$= 0.15 + 0.15 + 0.3 = 0.6$$ 1

ii) $\{B \geq 4\} = \{(i,j) \in S \mid i \in \{1,\dots,6\}, j \in \{4,\dots,6\}\}$

$$P(B \geq 4) = \sum_{j=4}^{6} \phi(j)$$

$$= 0.6 \qquad (1)$$

c) i) $\Pr(A \geq 4 \cup B \geq 4) =$

$$= \Pr(A \geq 4 \cap B < 4) + \Pr(A \geq 4 \cap B \geq 4) + \Pr(A < 4 \cap B \geq 4)$$

$$\Pr(A \geq 4 \cap B < 4) = (0.6)\cdot(0.4) = \Pr(A \geq 4)(1 - P(B \geq 4))$$

$$\Pr(A \geq 4 \cap B \geq 4) = (0.6)(0.6)$$

$$\Pr(A < 4 \cap B \geq 4) = (0.4)(0.6)$$

$$\Pr(A \geq 4 \cup B \geq 4) = 0.6(0.4 + 0.6 + 0.4)$$

$$= (0.6)(1.4)$$

$$= 0.84 \qquad \tfrac{1}{2}$$

$\therefore$ (A) Divide event into mutually disjoint events & use sum of probabilities law.

(B) Use independence of dice rolls for $\Pr(A \cap B) = \Pr(A)\Pr(B)$

(C) Compute $\Pr(B < 4) = 1 - \Pr(B \geq 4)$. $\tfrac{1}{2}$

C ii) $Pr(A \geq 4 \cap B \geq 4)$

$$= Pr(A \geq 4)\, Pr(B \geq 4) = (0.6)^2 = 0.36 \quad \frac{1}{2}$$

Independence of ~~events~~ die rolls ensures product of probabilities. $\frac{1}{2}$

C iii)

$$\begin{aligned} Pr(A + B = 8) &= Pr(A=2, B=6) & &= (0.15)(0.3) \\ &+ Pr(A=3, B=5) & &+ (0.15)(0.15) \\ &+ Pr(A=4, B=4) & &\quad (0.15)(0.15) \\ &+ Pr(A=5, B=3) & &\quad (0.15)(0.15) \\ &+ Pr(A=6, B=2) & &\quad (0.3)(0.15) \end{aligned}$$

$$= (0.15)(0.3 + 3(0.15) + 0.3)$$

$$= \cancel{0.151} \; 0.1575 \quad \frac{1}{2}$$

→ Disjoint events & sum over probabilities $\frac{1}{2}$

C iv)

$$Pr(A+B=8 \mid A \geq 4) = \frac{Pr(A+B=8, A \geq 4)}{Pr(A \geq 4)}$$

$$= \frac{(0.15)(0.15) + (0.15)(0.15) + (0.3)(0.15)}{0.6}$$

$$= 0.15 \quad \frac{1}{2}$$

→ Rule of conditional probability & then compute by sum of disjoint events. $\frac{1}{2}$

d i) $E[x] = \sum_{x=2}^{12} \phi(x)\, x$

$= 8$ take from table. 1

d ii) $E[(x-\bar{x})^2] = \sum_{x=2}^{12} \phi(x)(x-8)^2$

~~0.01(36) + 0.03(25) + 0.075(16) + 0.0975(9) + 0.15(4) + 0.1575(1)~~ 1

$= 6$

Easiest approach is list ~~variance~~ $(x-\bar{x})^2$ terms under table of ϕ_x + then do the calculator thing.

2.5

e)

i) Use a z-test because the variance of null hypothesis is known $\sigma^2_{ab} = 5.833$. 1

ii) $$Z = \frac{\bar{X} - \mu_{ab}}{\sigma_{ab}/\sqrt{n}} = \frac{7.7 - 7}{\sqrt{5.833}/\sqrt{50}}$$ 1

$$= 2.05$$

iii) $P = 0.020$ p-value for z-score 1

iv) The significance level $\alpha = 0.05$.

The alt. hypoth is two sided hence 1

we compare $P = 0.0207 < 0.025 = \frac{1}{2}\alpha$.

The P value is less than the significance level & hence ~~the test~~ does support the claim.

Q3

5.1

ai) This is justified by marginalisation of the joint probability

$$\sum_{C=\{W,D\}} \phi(T,C|R=1) = \phi(T|R=1)$$. (1)

aii) Bayes law.

$$\phi(T,C|R=1) = \frac{\phi(R=1|T,C)\,\phi(T,C)}{\phi(R=1)}$$ (1)

aiii) Conditional probability (1)

$$\phi(T,C) = \phi(T|C)\,\phi(C)$$

b) bi) $\phi(T=H|R=1)\,\phi(R=1) = \sum_{C=\{W,D\}} \phi(R=1|T,C)\,\phi(T|C)\,\phi(C)$

$$= (0.05)(0.4)(0.6) + (0.2)(0.7)(0.4)$$

$$= 0.068$$ (1/2)

3.2

b ii) $\phi(T=S \mid R=1)\,\phi(R=1) = \sum_{C=\{W,D\}} \phi(R=1 \mid T=S, C)\,\phi(T=S \mid C)\,\phi(C)$

$$= (0.2)(0.6)(0.6) + (0.05)(0.3)(0.4)$$

$$= 0.078.$$

1/2

b ii) $\Pr(T=H \mid R=1) = \dfrac{0.068}{0.068 + 0.078} = 0.466.$

1

b iii) I would recomend (T = W) wet weather tires.

1

c) $\phi(T=H \mid R=\{1,2,3\}) = \sum_{C=\{W,D\}} \dfrac{\phi(R=\{1,2,3\} \mid T=H, C)\,\phi(T=H \mid C)\,\phi(C)}{\phi(R=\{1,2,3\})}$

$$= (0.05 + 0.05 + 0.1)(0.4)(0.6) + (0.2 + 0.2 + 0.2)(0.7)(0.4) / \phi(R=\{1,2,3\})$$

$$= 0.216 / \phi(R=\{1,2,3\})$$

1/2

3.3

$$\phi(T=S \mid R=\{1,2,3\}) = \frac{1}{\phi(R=\{1,2,3\})} \sum_{C=\{W,D\}} \frac{\phi(R=\{1,2,3\} \mid T=S, C)\,\phi(T=S)\,\phi(C)}{\phi(R=\{1,2,3\})}$$

$$= \frac{1}{\phi(R=\{1,2,3\})} \left((0.2+0.2+0.2)(0.6)(0.6) + (0.05+0.05+0.1)(0.3)(0.4) \right)$$

$$= \frac{1}{\phi(R=\{1,2,3\})} (0.24) \qquad \frac{1}{2}$$

$$\phi(T=H \mid R=\{1,2,3\}) = \frac{0.216}{0.216+0.24} = 0.47 \qquad \frac{1}{2}$$

For $R=\{1,2,3\}$ choose Hard tires $\frac{1}{2}$

$T=H$

Q4)

a) i) $$\mu_\phi = \int_0^\infty x \lambda e^{-\lambda x} \, dx \qquad 1$$

ii) $$\cancel{\int_0^\infty x \lambda e^{-\lambda x} \, dx}$$

$$\cancel{\frac{d}{dx}\left(x^2 e^{-\lambda x}\right) = 2x e^{-\lambda x} - \lambda x^2 e^{-\lambda x}}$$

$$\frac{1}{2} \quad \int_0^\infty \frac{d}{dx}\left(x e^{-\lambda x}\right) = \int_0^\infty e^{-\lambda x} - \int_0^\infty \lambda x e^{-\lambda x} \, dx$$

$$\Rightarrow \quad \left[x e^{-\lambda x}\right]_{x=0}^{x=\infty} = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty - \mu_\phi$$

$$\frac{1}{2}$$

$$\Rightarrow \quad 0 - 0 = 0 + \frac{1}{\lambda} - \mu_\phi$$

$$\Rightarrow \quad \boxed{\mu_\phi = \frac{1}{\lambda}}$$

b)

$$\Phi_\mu(x) = \int_0^x \mu e^{-\mu\tau}\, d\tau \qquad \frac{1}{2}$$

$$= \left[-e^{-\mu\tau} \right]_0^x$$

$$= -e^{-\mu x} + 1$$

$$= 1 - e^{-\mu x} \qquad \frac{1}{2} \text{ QED.}$$

c)

i) Sampling dist

$$\phi_{\hat{\mu}}(x) = \phi_\mu(x) = \mu e^{-\mu x} \qquad 1/2$$

ii) ~~[illegible]~~

A single sample drawn from dist. ϕ_μ has distribution

$$\hat{\mu} = X_1 \sim \phi_\mu \qquad \frac{1}{2}$$

d). $x_1 = 10$

50% conf. lower band $\Rightarrow P = 0.25$

upper band $P = 0.75$

$\hat{\mu} = 10$

$\hat{\phi} = \phi_{1/\hat{\mu}} = \phi_{1/10}$

1/2

lower band x_l

$\Phi(x_l) = 0.25$

$\Rightarrow 1 - e^{-x_l/10} = 0.25$

$-10 \ln(0.75) = x_l$

$x_l = 2.88$

1/2 for the idea.

upper band:

$\Phi(x_u) = 0.75$

$1 - e^{-x_u/10} = 0.75$

$x_u = -10 \ln(0.25)$

$= 13.86$

$\text{Conf Int}_{50\%} = [2.88, 13.86]$

Q5 (5.1)

a i)

$$\phi(x_k = i \mid y_k = j) = \frac{\phi(y_k = j \mid x_k = i)\,\phi(x_k = i)}{\phi(y_k = j)}$$ ($\frac{1}{2}$)

$$= \frac{\phi(y_k = j \mid x_k = i)\ 1/2}{\sum_s \phi(y_k = j \mid x_k = s)\underbrace{\phi(x_k = s)}_{= 1/2}}$$

$$= \frac{\cancel{1/2}\ \phi(y_k = j \mid x_k = i)}{\cancel{1/2}\ \sum_s \phi(y_k = j \mid x_k = s)}$$

$$\sum_s \phi(y_k = 0 \mid x_k = s) = 0.6 + 0.4 = 1$$ ($\frac{1}{2}$)

$$\sum_s \phi(y_k = 1 \mid x_k = s) = 0.4 + 0.6 = 1$$

$$\Rightarrow \boxed{\phi(x_k = i \mid y_k = j) = \phi(y_k = j \mid x_k = i)}$$

ii) $\phi(x_k = 0 \mid y_k = 0) = 0.6$ ($\frac{1}{2}$)

$\phi(x_k = 1 \mid y_k = 0) = 0.4$ ($\frac{1}{2}$)

$\phi(x_k = 0 \mid y_k = 1) = 0.4$ ($\frac{1}{2}$)

$\phi(x_k = 1 \mid y_k = 1) = 0.6$ ($\frac{1}{2}$)

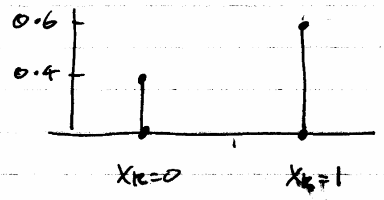
a-iii) $Pr(y_k \neq x_k) = \phi(y_k = 0 \mid x_k = 1) \cdot \phi(x_k = 1) + \phi(y_k = 1 \mid x_k = 0) \cdot \phi(x_k = 0)$ $\frac{1}{2}$

$$= \frac{1}{2}(0.4 + 0.4)$$

$$= 0.4. \quad \frac{1}{2}$$

b)

i)



1 or $Pr(X_k=0) = 0.4$

$Pr(X_k=1) = 0.6$

ii) $E[X_k] = (0.4)0 + (0.6)1 = 0.6$ $\frac{1}{2}$ $\frac{1}{2}$

iii) $E[T] = 1000\, E[X_k] = 600$ $\frac{1}{2}$

Each instance x_k is an indep event. $\frac{1}{2}$

5·3

c)

i) Prob of transmitting a false character

$$= Pr(X_1=0, X_2=1, X_3=1) + Pr(X_1=1, X_2=0, X_3=1) + Pr(X_1=1, X_2=1, X_3=0)$$

$$+ Pr(X_1=0, X_2=0, X_3=0)$$

$$= 0.6 \times (0.4)^2 + 0.6 \times 0.4^2 + 0.6 \times (0.4)^2$$

$$+ (0.4)^3$$

$$= 0.496$$

1

Call this event 'F $\rightarrow$ $Pr(F) = 0.352$

Prob of trans a valid character is (event V)

$$Pr(V) = 0.504$$

Cii)

for formulae

$\frac{1}{2}$

$$Pr(2 \text{ correct bits}) = Pr(X_1=1, X_2=1 \mid V)$$

$$= \frac{Pr(X_1=1, X_2=1, V)}{Pr(V)}$$

by inspection

$$= \frac{Pr(X_1=1, X_2=1, X_3=1)}{Pr(V)}$$

$$= \frac{(0.6)^3}{0.504} = 0.429$$

$\frac{1}{2}$

5.4

$$\Pr(1\text{ bit}\mid V) = \Pr(x_1=1, x_2=0 \mid V) + \Pr(x_1=0, x_2=1 \mid V)$$

$$= \frac{\Pr(x_1=1, x_2=0, x_3=0)}{\Pr(V)} + \frac{\Pr(x_1=0, x_2=1, x_3=0)}{\Pr(V)}$$

$$= \frac{(0.4)^2(0.6) + (0.4)^2(0.6)}{.504}$$

$$= 0.381$$

$\frac{1}{2}$

$$\Pr(0\text{ bits}\mid V) = \Pr(x_1=0, x_2=0 \mid V)$$

$$= \frac{\Pr(x_1=0, x_2=0, x_3=1)}{\Pr(V)}$$

$$= \frac{(0.4)^2\, 0.6}{\Pr(V)}$$

$$= 0.190$$

$\frac{1}{2}$

c iii) $$E[X] = \left(2\Pr(2\text{ correct bits}) + 1\Pr(1\text{ correct}) + 0\Pr(0\text{ correct})\right)/2$$ $\frac{1}{2}$

$$= \left(2(0.429) + (0.381) + 0(0.190)\right)/2$$

$$= 1.24/2 = 0.62$$ $\frac{1}{2}$

d i) $T' \leq E[X]\cdot\frac{2}{3}\cdot 1000 \leq 413$ $\frac{1}{2}$ $\boxed{T' < T}$

ii) Although less errors are made the overhead in transmission will reduce the data rate. In fact T' is smaller than $E[X]\,\frac{2}{3}\cdot 1000$ since some symbols are retransmitted! $\frac{1}{2}$